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On Integrable Conditions of Generalized Almost Complex Structures

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Abstract: Generalized complex geometry is a new kind of geometrical structure which contains complex and symplectic geometry as its special cases. This paper gives the equivalence between the integrable conditions of a generalized almost complex structure in big bracket formalism and those in the general framework.

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1 Introduction

Generalized geometry was created by Hitchin[1] originally as a way of characterizing special geometry in low dimensions, and has been further developed by Hitchin's students.

In [2], the integrable conditions under which a generalized almost complex structure becomes a generalized complex structure have been given in general way. The same question is discussed by using the big bracket formulism in supermanifold geometry by Kosmann-Schwarzbach and Rubtsov[3].

In this paper, we firstly recall some basic notions and facts about generalized complex structures, and then we devote to proving the equivalence between the integrable conditions of a generalized almost complex structure in different formalism.

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2 Integrable Conditions of Generalized Almost Complex Structures

Assume that M is a real manifold with dimension $2n$ and set $\mathbb{T}M = TM \oplus T^*M$. A generalized almost complex is an endomorphism $\mathcal{N} : \mathbb{T}M \longrightarrow \mathbb{T}M$ such that

(1) $\mathcal{N}$ is symplectic, i.e., $\langle \mathcal{N}e_1,\ e_2\rangle + \langle e_1,\ \mathcal{N}e_2\rangle = 0$ for $e_1, e_2 \in \Gamma(\mathbb{T}M)$;

(2) $\mathcal{N}$ is complex, i.e., $\mathcal{N}^2 = -\mathrm{Id}$,

where $\langle\ \cdot\ ,\ \cdot\ \rangle$ denotes the natural pairing given by $\langle X + \xi,\ Y + \eta\rangle = \eta(X) + \xi(Y)$ for $X + \xi, Y + \eta \in \Gamma(\mathbb{T}M)$.

Any generalized almost complex structure $\mathcal{N}$ may be presented by classical tensor fields as follows:

$$\mathcal{N} = \begin{pmatrix} N & \pi^\sharp \\ \sigma^\sharp & -N^* \end{pmatrix},$$

where $\pi \in \Gamma(\wedge^2 TM)$, $\sigma \in \Omega^2(M)$, and N is a $(1, 1)$-tensor field over M, $\pi^\sharp : T^*M \longrightarrow TM$ denotes a linear map defined by $\pi^\sharp(\xi) = i_\xi \pi = \pi(\xi,\ \cdot\ )$ for $\xi \in \Omega^1(M)$. Similarly, $\sigma^\sharp$ is a linear map defined by $\sigma^\sharp(X) = i_X\sigma$ for $X \in \Gamma(TM)$, N^* is the dual map of N. Clearly, $\mathcal{N}$ is symplectic if $\mathcal{N}$ is of the above form.

For any $e = X + \xi \in \Gamma(\mathbb{T}M)$, we have

$$\mathcal{N}e = \begin{pmatrix} N & \pi^\sharp \\ \sigma^\sharp & -N^* \end{pmatrix}\begin{pmatrix} X \\ \xi \end{pmatrix} = N(X) + \pi^\sharp(\xi) + \sigma^\sharp(X) - N^*\xi.$$

This structure is described by the big bracket formulism in [3]. The big bracket, denoted by $\{\ \cdot\ ,\ \cdot\ \}$, is an even graded bracket on the space $\mathcal{O}$ of functions on the cotangent bundle ΠT^*M, which is a supermanifold given by TM, more details are found in [4]–[5]. The action of $\mathcal{N}$ on $e \in \Gamma(\mathbb{T}M)$ can be expressed as

$$\begin{aligned}\{e,\ \mathcal{N}\} &\triangleq \{e,\ N + \pi + \sigma\} \\ &= \{X + \xi,\ N + \pi + \sigma\} \\ &= \{X, N\} + \{X,\ \sigma\} + \{\xi,\ N\} + \{\xi,\ \pi\} \\ &= N(X) + \sigma^\sharp(X) - N^*\xi + \pi^\sharp(\xi),\end{aligned}$$

that is, $\mathcal{N}e = \{e,\ \mathcal{N}\}$ for any $e \in \Gamma(\mathbb{T}M)$.

In this paper, we sometimes abbreviate $\mathcal{N} = \begin{pmatrix} N & \pi^\sharp \\ \sigma^\sharp & -N^* \end{pmatrix}$ to $\mathcal{N} = N + \pi + \sigma$ and omit the composition operation sign "$\circ$" when expressing the composition of two operators. For example, we write $\sigma^\sharp\pi^\sharp$ for $\sigma^\sharp \circ \pi^\sharp$. We adopt the convention in [3], $\pi^\sharp(\xi) = \{\xi, \pi\}$ and $\sigma^\sharp(X) = \{X, \sigma\}$.

By definition of the generalized almost complex structure, the endomorphism $\mathcal{N} = N + \pi + \sigma$ of $\mathbb{T}M$ is a generalized almost complex structure if and only if the following equalities hold:

(a) $N^2 + \pi^\sharp\sigma^\sharp = -\mathrm{Id}$;

(b) $N\pi^\sharp = \pi^\sharp N^*$;

(c) $\sigma^\sharp N = N^*\sigma^\sharp$.