COMMUNICATIONS IN MATHEMATICAL RESEARCH
32(2)(2016), 117–121

Unstabilized Self-amalgamation of a Heegaard Splitting along Disks

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Abstract: In this paper, we prove that a self-amalgamation of a strongly irreducible Heegaard splitting along disks is unstabilized.

Key words: Heegaard splitting, self-amalgamation, unstabilized
2010 MR subject classification: 57N10, 57M50
Document code: A
Article ID: 1674-5647(2016)02-0117-05
DOI: 10.13447/j.1674-5647.2016.02.04

1 Introduction

Let M be a compact connected orientable 3-manifold. If an embedded closed orientable surface S cuts M into two compression bodies V and W such that $\partial_+ V = S = \partial_+ W$, then $V \bigcup_S W$ is a Heegaard splitting of M and S is called a Heegaard surface. A Heegaard splitting $V \bigcup_S W$ for M is called to be stabilized if there are essential disks D in V and E in W such that $|D \bigcap E| = 1$. Let F be a closed connected embedded surface in M. If F is separating in M, F cuts M into two pieces M_1 and M_2 with $F_1 \subset \partial M_1$ and $F_2 \subset \partial M_2$, the two cutting sections of F, we say M is an amalgamation of M_1 and M_2 along F_1 and F_2; if F is non-separating in M, the manifold $\tilde{M}$ obtained by cutting M open along F is connected with the two cutting sections F_1, F_2 of F lying in $\partial \tilde{M}$, we say M is a self-amalgamation of $\tilde{M}$ along boundary components of F_1 and F_2, and call $F = F_1 = F_2$ in M the amalgamated surface.

Suppose that M is an amalgamation of M_1 and M_2 along F_1 and F_2, and $F = F_1 = F_2$ in M. For a given Heegaard splitting $M_i = V_i \bigcup_{S_i} W_i$ of M_i, $i = 1, 2$, Schultens[1] constructed a natural Heegaard splitting for M, which is called amalgamation of $V_1 \bigcup_{S_1} W_1$ and $V_2 \bigcup_{S_2} W_2$, refer to Section 2 for the definition.

So a natural question is when is an amalgamation of two unstabilized Heegaard splittings unstabilized? A well-known result is the Gordon conjecture: The connected sum of

Received date: Nov. 24, 2014.
Foundation item: The NSF (11101058, 11329101 and 11471151) of China and the Fundamental Research Funds (DUT14ZD208 and DUT14LK12) for the Central Universities.
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unstabilized Heegaard splittings is never stabilized (see [2], Problem 3.91). This was proved independently by Bachman[3] and by Qiu and Scharlemannin[4]. When the genus of the amalgamated surface is positive, there exist many counterexamples showing that an amalgamation of two unstabilized Heegaard splittings might be stabilized (see [5]–[8]). On the other hand, many sufficient conditions for an amalgamation of two unstabilized Heegaard splittings to be unstabilized are given, see [9]–[12], where the gluing maps are required to be complicated enough, and [13]–[15], where the the factor Heegaard splitings are of "high" distance.

Let M be a self-amalgamation of $\tilde{M}$ along boundary components F_1 and F_2 of $\partial\tilde{M}$. Given a Heegaard splitting $V'\bigcup_{S'}W'$ for $\tilde{M}$, there is amalgamated Heegaard splittiing for M, obtained by an analogous construction to that of Schulten's. It has been proved by Du and Qiu[16] that the self-amalgamation of a "high" distance Heegaard splitting is unstabilized. Recently, Zou *et al.*[17] proved that the self-amalgamation of a Heegaard splitting of distance at least 3 is unstabilized. In [18], we generalize the self-amalgamation of a Heegaard splitting to the case where the amalgamated surface could be with nonempty boundaries. And we proved that if the Heegaard splitting is strongly irreducible and annulus-busting, then any self-amalgamation of the Heegaard splitting along any essential subsurfaces is unstabilized.

In this paper, we consider a special case when the amalgamated surface is a disk. We prove that the self-amalgamation of a strongly irreducible Heegaard splitting along two disjoint disks is unstabilized. The article is organized as follows: in Section 2, we review some necessary preliminaries. The statement and proof of the main result is given in Section 3.

2 Preliminary

Let M be a compact orientable 3-manifold and F be a properly embedded surface in M. F is said to be compressible if either F is a 2-sphere which bounds a 3-ball or there is an essential simple closed curve on F which bounds a disk in M; otherwise, F is said to be incompressible. F is said to be essential if F is incompressible and no component of F is ∂-parallel in M. A simple closed curve in F is said to be essential if it is not contractible or ∂-parallel in F.

A Heegaard splitting $M = V\bigcup_S W$ is said to be reducible if there is an essential simple closed curve in S which bounds essential disks in both V and W; otherwise, it is irreducible. A Heegaard splitting $M = V\bigcup_S W$ is said to be weakly reducible if there are essential disks D in V and E in W such that $\partial D\bigcap\partial E = \emptyset$; otherwise, it is strongly irreducible (see [19]).

Assume that F is a closed surface in a compact orientable 3-manifold M which cuts M into two 3-manifolds M_1 and M_2. Let $V_i\bigcup_{S_i}W_i$ be a Heegaard splitting for M_i such that $F\subset\partial_-W_1, \partial_-V_2$ where $i = 1, 2$. Let $F\times[0, 1]$ be a regular neighborhood of F in M such that W_1 is obtained by attaching a collection of 1-handles to $F\times\left[0, \dfrac{1}{2}\right]$ along $F\times\{0\}$ and V_2 is obtained by attaching a collection of 1-handles to $F\times\left[\dfrac{1}{2}, 1\right]$ along $F\times\{1\}$. Denote