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Common Fixed Point Theorems and Q-property for Quasi-contractive Mappings under c-distance on TVS-valued Cone Metric Spaces without the Normality

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Abstract: Fixed point and common fixed point results for mappings satisfying quasi-contractive conditions expressed in the terms of c-distance on TVS-valued cone metric spaces (without the underlying cone which is not normal) are obtained, and P-property and Q-property for mappings in the terms of c-distance are discussed. Our results generalize and improve many known results.

Key words: TVS-valued cone metric space, quasi-contractive condition, common fixed point, P-property, Q-property

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1 Introduction and Preliminaries

Huang and Zhang[1] have introduced the concepts of cone metric spaces, where the set of real numbers is replaced by an order Banach space. Du *et al.*[2]–[10] discussed the existence problems of fixed points and common fixed points for mappings defined on TVS-valued cone metric spaces, where the order Banach space is replaced by a topological vector space.

Fixed point results in metric spaces with the so-called w-distance were obtained for the first by Kada *et al.*[11], where non-convex minimization problems were treated. Further results were given in [12]–[14]. The cone metric version of this notion (usually called a c-distance) was used in [15]–[16].

Wang and Guo[17] obtained a fixed point theorem for a mapping in normal cone metric

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space under some contractive condition expressed in the terms of c-distance, and Djordjević *et al.*[18] also obtained fixed point and common fixed point results for mappings in TVS-valued non-normal cone metric spaces under contractive condition expressed in the terms of c-distance. Those results generalize the corresponding results in [11]–[16] and the related results.

Here, we discuss the existence problems of fixed points and common fixed points for mappings on TVS-valued non-normal cone metric spaces under quasi-contractive condition expressed in the terms of c-distance and P-property and Q-property. The obtained results generalize the conclusions in [17]–[18] and other corresponding fixed point and common fixed point theorems.

Let $(E,\ \tau)$ be a topological vector space (TVS), P_0 a nonempty subset of E. P_0 is said to be a cone whenever

(i) P_0 is closed, nonempty and $P_0 \neq \{0\}$;

(ii) $ax + by \in P_0$ for all $x, y \in P_0$ and real numbers $a, b \geq 0$;

(iii) $P_0 \cap (-P_0) = \{0\}$.

In this paper, we always assume that the cone P_0 has a nonempty interior, i.e., $\mathrm{int}P_0 \neq \emptyset$ (such cones are called solid).

For a given cone $P_0 \subset E$, we define a partial ordering $\leq$ with respect to P_0 by $x \leq y$ if and only if $y - x \in P_0$. $x < y$ stands for $x \leq y$ and $x \neq y$, $x \ll y$ stands for $y - x \in \mathrm{int}P_0$.

A cone P_0 is called normal if there exists a real number $K > 0$ such that for all $x, y \in E$,

$$0 \leq x \leq y \Longrightarrow \|x\| \leq K\|y\|.$$

The least positive number K satisfying the above condition is called the normal constant of P_0.

It is known that a metric space is a normal cone metric space with normal constant $K = 1$.

Definition 1.1 *Let X be a nonempty set and E a TVS with solid cone P_0. Suppose that the mapping $d : X \times X \to E$ satisfies*

(1) $0 \leq d(x,\ y)$ *for all* $x, y \in X$, *and* $d(x,\ y) = 0$ *if and only if* $x = y$;

(2) $d(x,\ y) = d(y,\ x)$ *for all* $x, y \in X$;

(3) $d(x,\ y) \leq d(x,\ z) + d(z,\ y)$ *for all* $x, z, y \in X$.

Then d is called a TVS-valued cone metric on X, $(X,\ d)$ is called a TVS-valued cone metric space.

Definition 1.2 *Let $(X,\ d)$ be a TVS-valued cone metric space, $x \in X$ and $\{x_n\}_{n\in\mathbf{N}^+}$ a sequence in X. Then*

(1) $\{x_n\}$ *is a Cauchy sequence whenever for every* $c \in E$ *with* $0 \ll c$ *there exists an* $N \in \mathbf{N}^+$ *such that* $d(x_m,\ x_n) \ll c$ *for all* $n, m > N$;

(2) $\{x_n\}$ *converges to* x *whenever for every* $c \in E$ *with* $0 \ll c$, *there exists an* $N \in \mathbf{N}^+$ *such that* $d(x_n,\ x) \ll c$ *for all* $n > N$. *We denote this by* $x_n \to x$ *or* $\lim\limits_{n\to\infty} x_n = x$;

(3) $(X,\ d)$ *is called complete if every Cauchy sequence in* X *is convergent.*