Communications
in
Mathematical
Research
32(3)(2016), 259–271

On Skew Triangular Matrix Rings

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Communicated by Du Xian-kun

Abstract: Let α be a nonzero endomorphism of a ring R, n be a positive integer and $T_n(R,\alpha)$ be the skew triangular matrix ring. We show that some properties related to nilpotent elements of R are inherited by $T_n(R,\alpha)$. Meanwhile, we determine the strongly prime radical, generalized prime radical and Behrens radical of the ring $R[x;\alpha]/(x^n)$, where $R[x;\alpha]$ is the skew polynomial ring.

Key words: skew triangular matrix ring, skew polynomial ring, weak zip property, strongly prime radical, generalized prime radical

2010 MR subject classification: 16N20, 16S36

Document code: A

Article ID: 1674-5647(2016)03-0259-13

DOI: 10.13447/j.1674-5647.2016.03.08

1 Introduction

Throughout this paper, R denotes an associative ring with identity and α is a nonzero endomorphism of R. For a given ring R, we use $nil(R)$, $Nil_*(R)$, $Nil^*(R)$, $L\text{-}rad(R)$ and $J(R)$ to denote the set of all nilpotent elements, the prime radical, the upper nilradical, the Levitzki radical and the Jacobson radical of R, respectively. We denote by $R[x;\alpha]$ the skew polynomial ring, whose elements are the polynomials over R, the addition is defined as usual, and the multiplication subject to the relation $xr=\alpha(r)x$ for any $r\in R$. For a positive integer n, the skew triangular matrix ring is defined as

Received date: Oct. 3, 2015.

Foundation item: The NSF (11071097, 11101217) of China and the NSF (BK20141476) of Jiangsu Province of China.

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$$T_n(R,\alpha)=\left\{\begin{pmatrix} a_0 & a_1 & a_2 & \dots & a_{n-1}\\ 0 & a_0 & a_1 & \dots & a_{n-2}\\ 0 & 0 & a_0 & \dots & a_{n-3}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \dots & a_0 \end{pmatrix} \mid a_i\in R,\ i=0,1,\cdots,n-1\right\}$$

with addition pointwise and multiplication given by

$$\begin{pmatrix} a_0 & a_1 & a_2 & \dots & a_{n-1}\\ 0 & a_0 & a_1 & \dots & a_{n-2}\\ 0 & 0 & a_0 & \dots & a_{n-3}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \dots & a_0 \end{pmatrix}\begin{pmatrix} b_0 & b_1 & b_2 & \dots & b_{n-1}\\ 0 & b_0 & b_1 & \dots & b_{n-2}\\ 0 & 0 & b_0 & \dots & b_{n-3}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \dots & b_0 \end{pmatrix}=\begin{pmatrix} c_0 & c_1 & c_2 & \dots & c_{n-1}\\ 0 & c_0 & c_1 & \dots & c_{n-2}\\ 0 & 0 & c_0 & \dots & c_{n-3}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \dots & c_0 \end{pmatrix},$$

where

$$c_i=a_0\alpha^0(b_i)+a_1\alpha^1(b_{i-1})+\cdots+a_i\alpha^i(b_0),\qquad 0\le i\le n-1.$$

We denote elements of $T_n(R,\alpha)$ by $(a_0,\ a_1,\ \cdots,\ a_{n-1})$. It is easy to verify that the $\sigma: T_n(R,\alpha)\longrightarrow R[x;\alpha]/(x^n)$ defined by $\sigma(a_0,a_1,\cdots,a_{n-1})=a_0+a_1x+\cdots+a_{n-1}x+(x^n)$ is a ring isomorphism, where $a_i\in R$, $0\le i\le n-1$, (x^n) is the ideal generated by x^n.

The triangular matrix ring $T_n(R)$ and the quotient $R[x]/(x^n)$ of a polynomial ring $R[x]$ has attracted a lot of attention (see [1]–[3]). Nasr-Isfahani and Moussavi[4] discussed their right mininjective, right T-nilpotent and right Kasch property. In recent, Nasr-Isfahani[5] extended the study to the skew triangular matrix ring $T_n(R,\alpha)$ and gave their prime, primitive and maximal ideals. We continue in this paper investigate some properties of $T_n(R,\alpha)$ and determine the strongly prime radical, generalized prime radical and Behrens radicals of the quotient ring $R[x;\alpha]/(x^n)$.

2 Properties Related to Nilpotent Elements

Recall that a ring R is reduced if R has no nonzero nilpotent elements, R is an NI ring if $nil(R)=Nil^*(R)$, R is 2-primal if $nil(R)=Nil_*(R)$, R is weakly 2-primal if $nil(R)=L$-$rad(R)$, and R is locally 2-primal if each finite subset generates a 2-primal ring. A ring R is called nil-semicommutative if for every $a,b\in R$, $ab\in nil(R)$ implies $aRb\subseteq nil(R)$, and R is called weakly semicommutative if for any $a,b\in R$, $ab=0$ implies $aRb\subseteq nil(R)$. The following implications hold:

$$\text{reduced}\Rightarrow\text{2-primal}\Rightarrow\text{locally 2-primal}\Rightarrow\text{weakly 2-primal}\Rightarrow\text{NI}$$
$$\Rightarrow\text{nil-semicommutative}\Rightarrow\text{weakly semicommutative}.$$

In general, each of these implications is irreversible (see [6]).

Observe that $nil(T_n(R,\alpha))=(nil(R),R,\cdots,R)$, we have that a ring R is reduced if and only if

$$nil(R[x;\alpha]/(x^n))=Rx+\cdots+Rx^{n-1}+(x^n).$$