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Fock-Sobolev Spaces and Weighted Composition Operators among Them

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Abstract: We characterize the boundedness and compactness of weighted composition operators among some Fock-Sobolev spaces. We also estimate the norm and essential norm of these operators. Furthermore, we discuss the duality spaces of Fock-Sobolev spaces $\mathcal{F}_s^{p,m}$ when $0 < p < \infty$.

Key words: Fock-Sobolev space, dual space, weighted composition operator

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1 Introduction

Throughout this paper, let $\mathbf{C}^n$ be the complex n-space and $\mathrm{d}v$ be the usual volume measure on $\mathbf{C}^n$. For any points $z = (z_1, \cdots, z_n) \in \mathbf{C}^n$ and $w = (w_1, \cdots, w_n) \in \mathbf{C}^n$, we denote

$$\langle z, w\rangle = z\bar{w} = \sum_{i=1}^{n} z_1\bar{w}_1 + \cdots + z_n\bar{w}_n, \qquad |z| = \sqrt{\langle z, z\rangle}.$$

Suppose that $\alpha = (\alpha_1, \cdots, \alpha_n)$ is an n-tuple indices of non-negative integers. Write

$$\alpha! = \alpha_1! \cdots \alpha_n!, \quad |\alpha| = |\alpha_1| + \cdots + |\alpha_n|, \quad z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}, \quad \partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}.$$

For any $0 < p \le \infty$ and $s > 0$, take

$$L_s^p = \{f \text{ is a Lebesgue measurable function on } \mathbf{C}^n \mid f(w)\mathrm{e}^{-\frac{s}{2}|w|^2} \in L^p(\mathbf{C}^n, \mathrm{d}v)\}.$$

When $0 < p < \infty$, we norm the space L_s^p as

$$\|f\|_{L_s^p} = \left\{C_{n,p,s} \int_{\mathbf{C}^n} \left|f(w)\mathrm{e}^{-\frac{s}{2}|w|^2}\right|^p \mathrm{d}v(w)\right\}^{\frac{1}{p}},$$

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where $C_{n,p,s} = \left(\dfrac{ps}{2\pi}\right)^n$ is the normalizing constant such that $\|1\|_{L_s^p} = 1$. For $p = \infty$, the norm on L_s^p is

$$\|f\|_{L_s^\infty} = \sup_{w\in\mathbf{C}^n} \left\{|f(w)|\mathrm{e}^{-\frac{s}{2}|w|^2}\right\}.$$

For $0 < p \le \infty$, $s > 0$, set

$$\mathcal{F}_s^p = L_s^p \bigcap H(\mathbf{C}^n),$$

where $H(\mathbf{C}^n)$ is the analytic function space of $\mathbf{C}^n$. The space $\mathcal{F}_s^p$ is called Fock space. Denote by $\|\cdot\|_{p,s}$ the norm on $\mathcal{F}_s^p$. Suppose that m is a non-negative integer, $\alpha \in \mathbf{N}^n$, we write the Fock-Sobolev space as

$$\mathcal{F}_s^{p,m} = \left\{f \in H(\mathbf{C}^n)\Big| \sum_{|\alpha|\le m} \|\partial^\alpha f\|_{p,s} < \infty\right\}, \tag{1.1}$$

whose definition was firstly introduced in [1] in the case of $s = 0$. Furthermore, the norm on $\mathcal{F}_s^{p,m}$ is defined as

$$\|f\|_{p,m,s} := \sum_{|\alpha|\le m} \|\partial^\alpha f\|_{p,s}, \qquad f \in \mathcal{F}_s^{p,m}.$$

For convenience, we simply denote $f \lesssim g$ or $g \gtrsim f$ if there is a positive constant C such that $f \le Cg$, and $f \sim g$ if $f \lesssim g$ and $g \lesssim f$.

Motivated by some recent ideas by Cho and Zhu[1], we show the equivalence that $\|f\|_{p,m,s} \sim \||w|^m f\|_{p,s}$ in the next section.

Let u and φ be entire functions on $\mathbf{C}^n$. The weighted composition operator uC_φ is defined by $uC_\varphi f = u\cdot(f\circ\varphi)$ for any entire function f. When $u = 1$, the C_φ is called a composition operator.

As we known, there are plenty of results concern the boundedness, compactness and Schatten p-class for composition operators and weighted composition operators among several Banach spaces, as a consequence, the norms and essential norms of composition operators and weighted composition operators on these spaces are estimated. For instance, Shapiro[2] gave an equivalent description about the compactness of C_φ on Hardy spaces and weighted Bergman spaces, and estimates the essential norm of C_φ by using the angular derivative of its inducing map. Čučković and Zhao[3] showed that uC_φ is bounded on Bergman space $L_a^2(\mathbb{D})$ if and only if $B_\varphi(|u|^2)$ is bounded on $\mathbb{D}$ and uC_φ is compact if and only if $B_\varphi(|u|^2)$ vanishes to zero at the boundary of $\mathbb{D}$, where $\mathbb{D}$ is the open unit disk of the complex plane $\mathbb{C}$ and

$$B_\varphi(|u|^2)(z) = \int_{\mathbb{D}} \frac{(1-|a|^2)^2|u(z)|^2}{|1-\bar{a}\varphi(z)|^4}\mathrm{d}A(z)$$

is the φ-Berezin transform of $|u|^2$. Indeed, for various spaces, the problems of composition operators or weighted composition operators as well as their applications have attracted many other authors: by Roan[4] among Lipschitz spaces, by Smith[5] among Bergman and Hardy spaces, by Zhao[6] from Bloch type spaces to Hardy and Besov spaces, by Tjani[7] among Bloch spaces and Besov spaces, BMOA and VMOA.

Moreover, extensions of many of these conclusions to Fock-type spaces are also obtained. Fock-type spaces, are also called as Bargmann-type spaces or Segal Bargmann-type spaces,