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Subsurface 1-distance of the Handlebody

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Abstract: For a handlebody H with $\partial H = S$, let $F \subset S$ be an essential connected subsurface of S. Let $\mathcal{C}(S)$ be the curve complex of S, $\mathcal{AC}(F)$ be the arc and curve complex of F, $\mathcal{D}(H) \subset \mathcal{C}(S)$ be the disk complex of H and $\pi_F(\mathcal{D}(H)) \subset \mathcal{AC}(F)$ be the image of $\mathcal{D}(H)$ in $\mathcal{AC}(F)$. We introduce the definition of subsurface 1-distance between the 1-simplices of $\mathcal{AC}(F)$ and show that under some hypothesis, $\pi_F(\mathcal{D}(H))$ comes within subsurface 1-distance at most 4 of every 1-simplex of $\mathcal{AC}(F)$.

Key words: handlebody, curve complex, arc and curve complex, subsurface 1-distance

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1 Introduction

In 1981, Harvey[1] introduced the curve complex $\mathcal{C}(S)$ of a surface S which is a finite-dimensional simplicial complex, and it was intended to capture some properties of combinatorial topology of S. Hempel[2] applied it to study the distance of Heegaard splittings of 3-manifolds. See [3] for a survey that gives a good account of the history of the mathematics of the curve complex, continuing up to the recent advance.

Arc and curve complex of a surface F is another complex of F which can be known as the generalization of curve complex of F. Johnson[4] applied it to show that if the subsurface distance of a Heegaard splitting is large, then any other Heegaard surface also contains this subsurface. Li[5] proved that under some conditions, if the 3-manifold M is not a I-bundle of surface S, then the image of disk complex has diameter at most 6 in $\mathcal{AC}(S)$.

Distance of the Heegaard splittings of a 3-manifold is the distance between 0-simplex of curve complex $\mathcal{C}(S)$. We introduced the concept of a 2-path between two 1-simplices in $\mathcal{C}(S)$ and proved that $\mathcal{C}(S)$ is P^1-connected (see [6]). Using the 2-path, we defined the 1-distance

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between two 1-simplices in $\mathcal{C}(S)$.

In this article, we introduce the subsurface 1-distance between 1-simplices of $\mathcal{AC}(F)$ and discuss the upper bound of subsurface 1-distance under some hypothesis. In Section 2, we give some basic definitions. In Section 3, we introduce the subsurface 1-distance between 1-simplices of $\mathcal{AC}(F)$ and prove the main theorem.

2 Preliminaries

In this section we review some basic definitions. The definitions and terminologies not mentioned here are standard, see [7].

A simplicial complex consists of a family of vertices and a family of simplices. Simplices are non-empty finite sets of vertices, subject only to the following two conditions: a non-empty subset of a simplex σ is a simplex (which is called a face of σ); every vertex belongs to some simplex. Let $\sigma = \{v_0, v_1, \cdots, v_p\}$ be a simplex. p is called the dimension of σ, and is denoted by $\dim\sigma$, i.e., $\dim\sigma$ =the number of the vertices in it minus 1. A 1-dimensional simplex is also called an edge. For a simplex σ, we denote the set of vertices of σ by vet(σ).

Definition 2.1 *Let $S = S_{g,b}$ be a genus g surface with b boundary components. The curve complex $\mathcal{C}(S)$ is defined as follows: The vertices of $\mathcal{C}(S)$ are the isotopy classes of non-trivial circles on S. A simplex of $\mathcal{C}(S)$ is a set of vertices $\{\gamma_0, \gamma_1, \cdots, \gamma_p\}$ such that $\gamma_0 = \langle C_0\rangle$, $\gamma_1 = \langle C_1\rangle$, $\cdots$, $\gamma_p = \langle C_p\rangle$ for a collection of pairwise disjoint and non-parallel circles C_0, C_1, $\cdots$, C_p.*

Clearly, $\mathcal{C}(S_{g,b}) = \emptyset$ if $g = 0$ and $b = 0, 1, 2$ or 3; $\dim S_{1,0} = 0$; for the other cases of g and b, $\dim\mathcal{C}(S_{g,b}) = 3g - 4 + b$.

Definition 2.2 *Let H be a handlebody with $\partial H = S$. Let $\mathcal{D}(H)$ be the subcomplex of $\mathcal{C}(S)$ with each vertex of $\mathcal{D}(H)$ corresponding to a curve that bounds a compressing disk in H. The $\mathcal{D}(H)$ is called the disk complex of H.*

Definition 2.3 *Let $S = S_{g,b}$ be a genus g surface with b boundary components. The arc and curve complex $\mathcal{AC}(S)$ is defined as follows: The vertices of $\mathcal{AC}(S)$ are the isotopy classes of essential (non-peripheral) simple closed curves and properly embedded arcs on S. A simplex of $\mathcal{AC}(S)$ is a set of vertices $\{\gamma_0, \gamma_1, \cdots, \gamma_p\}$ such that $\gamma_0 = \langle C_0\rangle$, $\gamma_1 = \langle C_1\rangle$, $\cdots$, $\gamma_p = \langle C_p\rangle$ for a collection of pairwise disjoint and non-parallel essential circles or arcs C_0, C_1, $\cdots$, C_p.*

If S is an annulus, then there are no essential closed curves, and the isotopy classes of essential arcs should have taken rel endpoints.

In this article, we do not distinguish between the vertex of $\mathcal{C}(S)(\mathcal{AC}(S))$ and its representation element.

If F is a connected proper essential subsurface in S which is not an annulus, then there is a map

$$\pi_F : \mathcal{C}^0(S) \to \mathcal{AC}^0(F) \cup \{\phi\}$$