COMMUNICATIONS IN MATHEMATICAL RESEARCH
33(2)(2017), 121–128

Boundedness in Asymmetric Quasi-periodic Oscillations

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Communicated by Li Yong

Abstract: In the paper, by applying the method of main integration, we show the boundedness of the quasi-periodic second order differential equation $x'' + ax^+ - bx^- + \phi(x) = p(t)$, where $a \neq b$ are two positive constants and $\phi(s)$, $p(t)$ are real analytic functions. Moreover, the $p(t)$ is quasi-periodic coefficient, whose frequency vectors are Diophantine. The results we obtained also imply that, under some conditions, the quasi-periodic oscillator has the Lagrange stability.

Key words: boundedness, quasi-periodic, KAM theorem
2010 MR subject classification: 34C15, 70H08
Document code: A
Article ID: 1674-5647(2017)02-0121-08
DOI: 10.13447/j.1674-5647.2017.02.03

1 Introduction

In 1999, Fabry and Mawhin[1] suggested studying the boundedness of all the solutions for the following differential equation:

$$x'' + ax^+ - bx^- + \phi(x) = p(t), \tag{1.1}$$

where $a \neq b$ are two positive constants, $p(t)$ is a periodic function, $x^+ = \max\{x, 0\}$ and $x^- = \max\{-x, 0\}$. Dancer[2] studied the periodic and Dirichlet boundary value problems for (1.1). Many authors are also concerned with the Lagrange stability of this equation. The first boundedness result for (1.1) is due to Ortega. In [3], Ortega obtained the stability for

$$x'' + ax^+ - bx^- = 1 + p(t) \tag{1.2}$$

under the assumption that the periodic function $p(t) \in C^4(\mathbf{R})$ is sufficiently small. Adding the condition $\dfrac{1}{\sqrt{a}} + \dfrac{1}{\sqrt{b}} \in Q$ and some other reasonable assumptions, Liu[4] also obtained

Received date: Aug. 12, 2015.
Foundation item: The NSF (201442137-26) of Xinjiang.
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the stability for (1.2). More related results can be found in [5]–[6] and the references therein. Recently, the boundedness and stability of quasi-periodic equation has been drawing more and more attention such as [7]–[10]. In this paper, we make some attempts for (1.1), where $p(t)$ is quasi-periodic and $\phi(x)$ is real analytic with the limits

$$\lim_{x\to\pm\infty}\phi(x)=\phi(\pm\infty) \tag{1.3}$$

are finite.

Firstly, we introduce some basic notations, properties and lemmas which are used in this paper.

Definition 1.1 [7] *A function $f:\mathbf{R}\to\mathbf{R}$ is called real analytic quasi-periodic with frequencies μ if it can be represented as a Fourier series of the type $f(t)=\sum\limits_k f_k\mathrm{e}^{i\langle k,\,\mu\rangle t}$, where*

$$k=(k_1,k_2,\cdots,k_m),\quad \mu=(\mu_1,\mu_2,\cdots,\mu_m),\quad \langle k,\,\mu\rangle=\sum k_j\mu_j.$$

Definition 1.2 [7] *Let $Q_r(\mu)\subset Q(\mu)$ be the set of the real analytic function f which is 2π periodic in each variables and bounded in $\varPi_r=\{(\theta_1,\theta_2,\cdots,\theta_m)\in\mathbf{C}^m:|\mathrm{Im}\,\theta_j|\le r\}$ for some $r>0$ with the supermum norm $\|f\|_r=\sup\limits_{\theta\in\varPi_r}|F(\theta)|$.*

Lemma 1.1 [7] *The following statements are true*:

(i) *Let $f(t)$, $g(t)\in Q(\mu)$. Then $g(t+f(t))\in Q(\mu)$.*

(ii) *Suppose that $|\langle k,\,\mu\rangle|\ge\dfrac{c}{|k|^\sigma}$. Let $h(t)\in Q(\mu)$ and $\tau=\alpha t+h(t)$ with $\alpha+h'>0$. Then the inverse relation is given by $t=\alpha^{-1}\tau+h_1(\tau)$ and $h_1(\tau)\in Q(\mu/\alpha)$.*

Sun[11] considered the map

$$M_\delta:\begin{cases} x_1=x+\alpha+\delta L(x,y)+\delta f(x,y,\delta),\\ y_1=y+\alpha+\delta M(x,y)+\delta g(x,y,\delta),\end{cases}$$

where L, M, f and g are quasi-periodic in x with the frequency μ and real analytic in the complex neighborhood of $\mathbf{R}\times[a,\,b]$. Assume that $f(x,y,0)=g(x,y,0)=0$ and the μ satisfies the Diophantine condition:

$$|\langle k,\,\mu\rangle|\ge\frac{\gamma}{|k|^\sigma},\qquad \gamma,\sigma>0 \tag{1.4}$$

for all integer $k=(k_1,k_2,\cdots,k_m)\ne0$. Furthermore, suppose that the map M_δ has the intersection property, that is, for every Jordan curve $y=\varphi(x)$ which is homotopic to the line satisfies $M_\delta(\varGamma)\cap\varGamma\ne\emptyset$. Sun obtained the following two theorems.

Theorem 1.1([11], Theorem 2.3) *If μ_1, μ_2, $\cdots$, μ_m and $2\pi/\alpha$ are rational independent and*

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T\frac{\partial L}{\partial y}\mathrm{d}x\ne0,\qquad y\in[a,\,b],$$

then there exists $\varDelta>0$ such that if $0<\delta<\varDelta$, the map M_δ has an invariant curve $y=\phi(x)$, where $\phi(x)$ is real analytic and quasi-periodic with the frequency μ.