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A New Characterization on g-frames in Hilbert C^*-Modules

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Abstract: In this note, we establish a new characterization on g-frames in Hilbert C^*-modules from the operator-theoretic point of view, with which we provide a correction to one result recently obtained by Yao (Yao X Y. Some properties of g-frames in Hilbert C^*-modules (in Chinese). *Acta Math. Sinica*, 2011, **54**(1): 1–8.).

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1 Introduction

The frames for a Hilbert space were introduced in the paper [1] by Duffin and Schaeffer from 1952, when they were used to study some deep problems in nonharmonic Fourier series. The importance of frames was not realized until 1986 when Daubechies *et al.*[2] found a fundamental new application, to wavelet and window Fourier transform. Since then, frames have become the focus of active research, both in theory and in applications, such as the characterization of function spaces, digital signal processing, scientific computations, etc.

The theory of frames in Hilbert spaces was rapidly extended and various generalizations of frame concept were developed. Among them, g-frames, proposed by Sun[3], include many other generalizations of frames, e.g., frames of subspaces (see [4]), oblique frames (see [5]), pseudo-frames (see [6]) and outer frames (see [7]), etc.

The frames and g-frames for Hilbert spaces have natural analogues for Hilbert C^*-modules (see [8] and [9]). Although Hilbert C^*-modules are generalizations of Hilbert spaces, there are many essential differences between them because of the complexity of

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the C^*-algebras involved in the Hilbert C^*-modules and the fact that some useful techniques available in Hilbert spaces are either absent or unknown in Hilbert C^*-modules. This suggests that the generalization of frame theory from Hilbert spaces to Hilbert C^*-modules is not a trivial task. The properties of g-frames in Hilbert C^*-modules were further investigated in [10]–[12].

In this paper, we study the equivalent characterization of g-frames in Hilbert C^*-modules. The motivation derives from an observation on a result obtained by Yao[11], which can be expressed as follows: an adjointable operator preserves g-frames in Hilbert C^*-modules if and only if it is invertible. However, a counterexample (see Example 3.1) shows that the "only if" part of the result is not true. In Section 3 of the present paper, we give a new characterization of g-frames in Hilbert C^*-modules in terms of operators (see Theorem 3.1) so that her result can be corrected (see Theorem 3.2).

2 Preliminaries

In the following we briefly recall some definitions and basic properties of operators and g-frames in Hilbert C^*-modules.

We first give some notations. Throughout this paper, the symbols $\mathbb{J}$ and $\mathcal{A}$ refer, respectively, to a finite or countable index set and a unital C^*-algebra. $\mathscr{H}$, $\mathscr{K}$ and $\mathscr{K}_j$ (for each $j \in \mathbb{J}$) are Hilbert C^*-modules over $\mathcal{A}$ (or simply, Hilbert $\mathcal{A}$-modules). We denote by $\mathrm{End}^*_{\mathcal{A}}(\mathscr{H},\ \mathscr{K})$ the set of all adjointable operators from $\mathscr{H}$ to $\mathscr{K}$, and $\mathrm{End}^*_{\mathcal{A}}(\mathscr{H},\ \mathscr{H})$ is abbreviated to $\mathrm{End}^*_{\mathcal{A}}(\mathscr{H})$. For $T \in \mathrm{End}^*_{\mathcal{A}}(\mathscr{H},\ \mathscr{K})$, the notations $\mathscr{R}(T)$ and $\mathscr{N}(T)$ are reserved respectively for the range and the null space of T.

Definition 2.1 *For each $j \in \mathbb{J}$, let $\Lambda_j \in \mathrm{End}^*_{\mathcal{A}}(\mathscr{H},\ \mathscr{K}_j)$. Then we call $\{\Lambda_j\}_{j\in\mathbb{J}}$ a g-frame for $\mathscr{H}$ with respect to $\{\mathscr{K}_j\}_{j\in\mathbb{J}}$, if there exist two constants $C, D > 0$ such that*

$$C\langle f,\ f\rangle \leq \sum_{j\in\mathbb{J}} \langle \Lambda_j f,\ \Lambda_j f\rangle \leq D\langle f,\ f\rangle, \qquad f \in \mathscr{H}. \tag{2.1}$$

We call C, D the lower and upper g-frame bounds, respectively. The g-frame $\{\Lambda_j\}_{j\in\mathbb{J}}$ is said to be λ-tight if $C = D = \lambda$, and said to be Parseval if $C = D = 1$. The sequence $\{\Lambda_j\}_{j\in\mathbb{J}}$ is called a g-Bessel sequence with bound D if we only require the right hand inequality of (2.1).

For each g-Bessel sequence $\{\Lambda_j \in \mathrm{End}^*_{\mathcal{A}}(\mathscr{H},\ \mathscr{K}_j)\}_{j\in\mathbb{J}}$ for $\mathscr{H}$ with respect to $\{\mathscr{K}_j\}_{j\in\mathbb{J}}$, we define the Hilbert C^*-module over $\mathcal{A}$ associated with $\{\Lambda_j\}_{j\in\mathbb{J}}$ by

$$\ell^2(\{\mathscr{K}_j\}_{j\in\mathbb{J}}) = \left\{\{g_j\}_{j\in\mathbb{J}} : g_j \in \mathscr{K}_j,\ \sum_{j\in\mathbb{J}} \langle g_j,\ g_j\rangle \text{ converges in } \|\ \cdot\ \|\right\},$$

and with the $\mathcal{A}$-valued inner product given by

$$\langle \{f_j\}_{j\in\mathbb{J}},\ \{g_j\}_{j\in\mathbb{J}}\rangle = \sum_{j\in\mathbb{J}} \langle f_j,\ g_j\rangle. \tag{2.2}$$

Definition 2.2 *Let $\{\Lambda_j \in \mathrm{End}^*_{\mathcal{A}}(\mathscr{H},\ \mathscr{K}_j)\}_{j\in\mathbb{J}}$ be a g-frame for $\mathscr{H}$ with respect to $\{\mathscr{K}_j\}_{j\in\mathbb{J}}$. The synthesis operator $U : \ell^2(\{\mathscr{K}_j\}_{j\in\mathbb{J}}) \to \mathscr{H}$ is defined by*