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Growth of Solutions to Higher Order Differential Equation with Meromorphic Coefficients

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Abstract: In this paper, we study the growth of solutions of higher order differential equation with meromorphic coefficients, and find some conditions which guarantee that its every nontrivial solution is of infinite order.

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1 Introduction and Main Results

In this paper, we adopt the standard notations and the fundamental results of the Nevanlinna's value distribution theory of meromorphic functions as explained in [1]–[3]. In addition, we use notations $\sigma(f)$ and $\sigma_2(f)$ to denote the order and the hyper-order of a meromorphic function f, $\lambda(f)$ and $\bar{\lambda}(f)$ to denote the exponent of convergence of the zero-sequence and the distinct zero-sequence of f respectively.

Consider the differential equation

$$f'' + \mathrm{e}^{-z} f' + B(z) f = 0, \tag{1.1}$$

where $B(z)$ is an entire function. Frei[4] proved that if B is a constant, then (1.1) has a solution of finite order if and only if $B = -n^2$, where n is an integer. If B is a nonconstant polynomial or B is transcendental of $\sigma(B) \neq 1$, Ozawa[5], Langely[6] and Gundersen[7] proved that all nontrivial solutions of (1.1) have infinite order. Thus there arises the following question: If B is transcendental of $\sigma(B) = 1$, when does (1.1) have a solution of infinite

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order? Lots of work have been done on this direction (see [8]–[12], etc.).

In [8], Chen proved the following result, which improved results of Frei[4], Ozawa[5], Langley[6] and Gundersen[7].

Theorem A[8] *Let $A_0(z)$, $A_1(z)$ be entire functions of orders less than 1 and a, b be two complex numbers satisfying $ab \neq 0$ and $a \neq b$. Then every solution $f(\not\equiv 0)$ of the equation*

$$f'' + A_1(z)\mathrm{e}^{az} f' + A_0(z)\mathrm{e}^{bz} f = 0$$

is of infinite order.

Recently, the authors in [9] extended Theorem A to some higher order differential equations

$$f^{(k)} + \sum_{j=1}^{k-1} (B_j \mathrm{e}^{b_j z} + D_j \mathrm{e}^{d_j z}) f^{(j)} + (A_1 \mathrm{e}^{a_1 z} + A_2 \mathrm{e}^{a_2 z}) f = 0 \tag{1.2}$$

and proved the following result.

Theorem B[9] *Let $k \geq 2$ be an integer, A_1, A_2, B_j, D_j $(j = 1, \cdots, k-1)$ be nonzero entire functions of orders less than 1 and a_1, a_2, b_j, d_j $(j = 1, \cdots, k-1)$ be nonzero complex numbers such that $a_1 \neq a_2$ and $b_j < 0$. Suppose that there exists $\alpha_j \in (0, 1)$, $\beta_j \in (0, 1)$ such that $d_j = \alpha_j a_1 + \beta_j a_2$. Set $\alpha = \max\limits_{1 \leq j \leq k-1} \{\alpha_j\}$, $\beta = \max\limits_{1 \leq j \leq k-1} \{\beta_j\}$, $b = \min\limits_{1 \leq j \leq k-1} \{b_j\}$. If*

(1) $\arg a_1 \neq \pi$ *and* $\arg a_1 \neq \arg a_2$; *or*

(2) $\arg a_1 \neq \pi$, $\arg a_1 = \arg a_2$ *and*

  (i) $|a_2| > \dfrac{|a_1|}{1-\beta}$ *or*

  (ii) $|a_2| < (1-\alpha)|a_1|$; *or*

(3) $a_1 < 0$ *and* $\arg a_1 \neq \arg a_2$; *or*

(4) (i) $(1-\beta)a_2 - b < a_1 < 0$, $a_2 < \dfrac{b}{1-\beta}$ *or*

  (ii) $a_1 < \dfrac{a_2 + b}{1-\alpha}$, $a_2 < 0$,

then every solution $f(\not\equiv 0)$ of (1.2) satisfies $\sigma(f) = +\infty$ and $\sigma_2(f) = 1$.

In Theorems A and B, the authors considered the case that all coefficients of the above equations are entire functions. In this paper, we are concerned with the more general case. In fact, we consider the following differential equation

$$f^{(k)} + \sum_{j=1}^{k-1} (A_j \mathrm{e}^{P_j(z)} + B_j \mathrm{e}^{Q_j(z)}) f^{(j)} + (A_0 \mathrm{e}^{P_0(z)} + B_0 \mathrm{e}^{Q_0(z)}) f = 0 \tag{1.3}$$

with meromorphic coefficients, and obtain the following results.

Theorem 1.1 *Let $P_j(z) = a_{jn}z^n + \cdots + a_{j0}$, $Q_j(z) = b_{jn}z^n + \cdots + b_{j0}$ $(j = 0, \cdots, k-1)$ be polynomials with degree $n(\geq 1)$ such that $a_{0n} \neq b_{0n}$ and $a_{jn} < 0$, $b_{jn} = \alpha_j a_{0n} + \beta_j b_{0n}$ $(0 < \alpha_j < 1, 0 < \beta_j < 1, j = 1, \cdots, k-1)$. And let A_j, B_j $(j = 0, \cdots, k-1)$ be meromorphic functions with orders less than n and $A_0 B_0 \not\equiv 0$. If one of the following cases occurs:*

(i) $\arg a_{0n} \neq \arg b_{0n}$;