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Uniqueness in Determining a Ball with a Single Incoming Wave

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Abstract: We prove that a ball with the impedance boundary condition is uniquely determined by the far-field pattern corresponding to an incident plane wave at one given wavenumber and one given incident direction. In the uniqueness proof, the impedance parameter in the impedance boundary condition is unknown.

Key words: inverse obstacle scattering, uniqueness, ball, impedance boundary condition, Helmholtz equation

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1 Introduction

In this paper, we mainly consider the uniqueness of an inverse acoustic obstacle scattering problem. Mathematically, the scattering of time-harmonic acoustic waves by an obstacle embedded in a homogeneous medium can be formulated as the following acoustic obstacle scattering problem: Let $G \subset \mathbf{R}^3$ be an impenetrable bounded obstacle with a C^2-smooth boundary surface and an unbounded and open connected complement, given an incident field $u^i(x) = \mathrm{e}^{\mathrm{i}kx\cdot d}$ with wavenumber $k > 0$ and incident direction d, find the scattered field u^s and the total field $u = u^i + u^s$ such that

$$\begin{cases} \Delta u + k^2 u = 0 & \text{in } \mathbf{R}^3 \setminus \bar{G}, \\ B_u(x) = 0 & \text{on } \partial G, \\ \lim\limits_{r\to\infty} r\left(\dfrac{\partial u^s}{\partial \nu} - \mathrm{i}ku^s\right) = 0, & r = |x|. \end{cases} \tag{1.1}$$

where ν is the exterior unit normal vector to ∂G. The operator B_u in (1.1) defines the boundary condition, the frequently occurring boundary condition is the impedance boundary

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condition

$$B_u(x) := \frac{\partial u}{\partial \nu}(x) + \mathrm{i}\lambda u(x) = 0 \qquad \text{on } \partial G$$

for impedance obstacle with some real-valued impedance parameter $\lambda \geq 0$ on ∂G. The Dirichlet boundary condition for sound-soft obstacle is the impedance boundary condition with $\lambda = \infty$, and the Neumann boundary condition for sound-hard obstacle is the impedance boundary condition with $\lambda = 0$.

It is known that the scattered field u^s has an asymptotic behavior of the form

$$u^s(x) = \frac{\mathrm{e}^{\mathrm{i}k|x|}}{|x|}\left\{u_\infty(\hat{x}) + O\left(\frac{1}{|x|}\right)\right\}, \qquad |x| \to \infty$$

uniformly in all directions $\hat{x} = x/|x|$, where the function u_∞ defined on the unit sphere $\mathbf{S}^2$ is known as the far-field pattern of u^s.

The inverse acoustic obstacle scattering problem is to determine the shape and location of the scatterer G from a knowledge of the far-field pattern u_∞. This problem is important in many applications, such as radar/sonar applications, geophysical exploration and medical imaging. One of the fundamental issues for the inverse problem is the uniqueness, i.e., whether the scatterer can be uniquely determined by the given far field pattern.

Most of the existing works on the uniqueness problem identify the obstacle with the far-field patterns from infinitely or countably many incident plane waves (see [1], [2] and references therein). Nevertheless, there is a widespread belief that one can establish the uniqueness in determining an obstacle G by the far-field pattern from a single incident plane wave (see [2]). Some reconstruction schemes with a single incident wave have been developed (see, e.g., [3]–[5]), while the uniqueness still remains largely open in the literature, though significant efforts have been devoted to such a study (see [6]–[12]).

Assuming the scatterer to be a ball may be a nice try to establish the uniqueness for one incident wave, because any C^2-smooth obstacle is isomorphic to a ball. With the far-field pattern from a single incident plane wave, Liu[9] established a uniqueness result for the sound-soft ball and Yun[10] established a uniqueness result for the sound-hard ball. [11] and [12] even showed that a sound-soft/sound-hard ball can be uniquely determined by some far-field data measured at some fixed spots corresponding to a single incident plane wave. Could uniqueness be established for impedance ball? In the present paper, we make a step towards this direction. We prove that an impedance ball without the impedance parameter provided can be uniquely determined with only the far-field pattern from just a single incident plane wave. As a matter of fact, the uniqueness result in the present work includes the uniqueness results in [9] and [10]. To our knowledge, this is the first result of such kind for the impedance ball. The proof is inspired by the proof of Theorem 5.4 in [1], it is based on a translation property of the far field pattern, and also the series expansion of the scattered wave fields for an impedance ball. Finally, we wish to remark that we are only concerned with the three-dimensional case, but all the result obtained can be straightforwardly extended to the two-dimensional case.