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Two Bijections on Weighted Motzkin Paths

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Abstract: In this paper, we provide a bijection between the set of underdiagonal lattice paths of length n and the set of $(2, 2)$-Motzkin paths of length n. Besides, we generalize the bijection of Shapiro and Wang (Shapiro L W, Wang C J. A bijection between 3-Motzkin paths and Schröder paths with no peak at odd height. *J. Integer Seq.*, 2009, **12**: Article 09.3.2.) to a bijection between k-Motzkin paths and $(k-2)$-Schröder paths with no horizontal step at even height. It is interesting that the second bijection is a generalization of the well-known bijection between Dyck paths and 2-Motzkin paths.

Key words: underdiagonal lattice path, $(2, 2)$-Motzkin path, k-Motzkin path, $(k-2)$-Schröder path
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1 Introduction

For any positive integer n, let P denote an underdiagonal lattice path of length n in the xy-plane that:

(1) consists of right steps $R = (1, 0)$, upward steps $W = (1, 2)$ and vertical steps $V = (0, 1)$;

(2) begins at $(0, 0)$ and terminates at (n, k), for $0 \leqslant k \leqslant n$;

(3) never rises above the line $y = x$.

The length of underdiagonal lattice path P, denoted by $length(P)$, is equal to n, whose last step reaches line $x = n$. Fig. 1.1 is an example of an underdiagonal lattice path of length 8.

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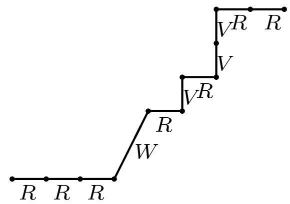

Fig. 1.1 An underdiagonal lattice path of length 8

Let $\mathcal{P}_n$ denote the set of underdiagonal lattice paths of length n, and p_n denote the number of $\mathcal{P}_n$ which is listed as entry A071356 in OEIS (see [1]), and its first few items are 1, 2, 6, 20, 72, 272, 1064, $\cdots$ See [2] for more information.

A Motzkin path of length n is defined as a lattice path from $(0, 0)$ to $(n, 0)$ consisting of up steps $U = (1, 1)$, horizontal steps $H = (1, 0)$ and down steps $D = (1, -1)$ that does not go below the x-axis. A (k, t)-Motzkin path is a Motzkin path that each horizontal step is colored with one of the k colors 1, 2, $\cdots$, k, and each down step is colored with one of the t colors 1, 2, $\cdots$, t. While $t = 1$, we call it a k-Motzkin path. A bijection between noncrossing linked partitions and large $(3, 2)$-Motzkin paths was given by Chen and Wang[3]. The properties of $(3, 2)$-Motzkin paths have been extensively studied by Woan[4],[5]. Recently, several papers on the combinatorics of Motzkin paths have been published (see [6]–[9]).

Let $\mathcal{M}_n$ denote the set of $(2, 2)$-Motzkin paths of length n, and let m_n denote the number of $\mathcal{M}_n$. We deduced that the generating function of m_n is $\dfrac{1 - 2x - \sqrt{(2x - 1)^2 - 8x^2}}{4x^2}$, and found that m_n fits the entry A071356 in OEIS (see [1]). In this paper, we shall construct a one-to-one correspondence between the set of underdiagonal lattice paths of length n and the set of $(2, 2)$-Motzkin paths of length n.

A Schröder path of length $2n$ is a lattice path from $(0, 0)$ to $(2n, 0)$ that does not go below the x-axis and consists of up steps $u = (1, 1)$, horizontal steps $h = (2, 0)$ and down steps $d = (1, -1)$. A k-Schröder path is a Schröder path whose horizontal steps could be colored by one of k colors. Yan[10] provided a one-to-one correspondence between $(2, 3)$-Motzkin paths and Schröder paths for the purpose of enumerating the UDD and LD subsequences. A bijection between 3-Motzkin paths of length $n - 1$ and Schröder paths of length $2n$ with no peak at odd height was given by Shapiro and Wang[11]. In this paper, we generalize it to a bijection between k-Motzkin paths of length n and $(k - 2)$-Schröder paths of length $2n + 2$ with no horizontal step at even height. Restricting $(k - 2)$-Schröder paths to Dyck paths, we get the well-known bijection between 2-Motzkin paths and Dyck paths. One can see [12] to learn more about the bijection.

2 Underdiagonal Lattice Paths and $(2, 2)$-Motzkin Paths

In this section, we aim to construct a bijection between the set of underdiagonal lattice paths of length n and the set of $(2, 2)$-Motzkin paths of length n, and present some interesting