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Monomial Derivations without Darboux Polynomials

Li Jian-tao
(*School of Mathematics, Liaoning University, Shenyang, 110036*)

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Abstract: In this paper, it is proved that a monomial derivation d of $k[x, y, z]$ has no Darboux polynomials if and only if d is a strict derivation with a trivial ring of constants, and we give the specific conditions when it has no Darboux polynomials.

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1 Introduction

Throughout this paper, let $k[X] = k[x_1, x_2, \cdots, x_n]$ denote the polynomial ring over a field k of characteristic 0.

A derivation $d = f_1\dfrac{\partial}{\partial x_1} + \cdots + f_n\dfrac{\partial}{\partial x_n}$ of $k[X]$ is said to be a monomial derivation if each f_i is a monomial in $k[X]$. By a Darboux polynomial of d we mean a polynomial $F \in k[X]$ such that $F \notin k$ and $d(F) = \varLambda F$ for some $\varLambda \in k[X]$, and the polynomial $\varLambda$ is called a cofactor of the Darboux polynomial F. The aim of this paper is to describe monomial derivations of $k[x, y, z]$ without Darboux polynomials.

Derivations and Darboux polynomials are useful algebraic methods to study the polynomial or the rational differential systems. If we associate a polynomial differential system

$$\frac{\mathrm{d}}{\mathrm{d}t}x_i = f_i, \qquad i = 1, 2, \cdots, n$$

with a derivation

$$d = f_1\frac{\partial}{\partial x_1} + \cdots + f_n\frac{\partial}{\partial x_n},$$

then the existence of the Darboux polynomials is a necessary condition for the system to have a first integral (see [1]–[3]). It is of interest to know whether Darboux polynomials exist

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E-mail address: jtlimath@qq.com (Li J T).

for any derivation of $k[X]$. However, in general, this problem is very difficult. In $k[x_1, x_2]$, d is a derivation without Darboux polynomials if and only if d is a simple derivation, and only some sporadic examples of derivations without Darboux polynomials are known (see [4]–[6]). For $n \geq 3$, the most famous example of derivations without Darboux polynomials is the Jouanolou derivation defined by $d(x) = y^s$, $d(y) = z^s$ and $d(z) = x^s$, $s \geq 2$ (see [7]–[9]).

If a derivation d has no Darboux polynomials, then the ring of constants of d, denoted by $k[X]^d$, must be trivial, that is, $k[X]^d = k$. In 2006, Nowicki and Zieliński[10] gave a full description of all monomial derivations of the rational function field $k(X)$ with trivial field of constants in two or three variables. They also proved that a generalized Jouanolou derivation

$$d = y^p \frac{\partial}{\partial x} + z^q \frac{\partial}{\partial y} + x^r \frac{\partial}{\partial z}, \qquad p,\ q,\ r \in \mathbf{Z},$$

has a trivial field of constants if and only if $pqr \geq 2$. In 2008, Moulin Ollagnier and Nowicki[11] presented several new examples of homogeneous monomial derivations without Darboux polynomials of $k[x, y, z]$, in which case

$$d = f_1 \frac{\partial}{\partial x} + f_2 \frac{\partial}{\partial y} + f_3 \frac{\partial}{\partial z},$$

and f_1, f_2 and f_3 are homogeneous monomials of the same degree $s \leq 4$. In 2011, Moulin Ollagnier and Nowicki[12] proved that a strict monomial derivation d of $k[x, y, z]$ has no Darboux polynomials if and only if $k(x, y, z)^d = k$.

In this paper, we show that a monomial derivation d of $k[x, y, z]$ has no Darboux polynomials if and only if d is strict and has a trivial ring of constants, that is, $k[x, y, z]^d = k$. It should be noted that the condition $k[x, y, z]^d = k$ cannot imply $k(x, y, z)^d = k$. Even if a derivation d has a trivial ring of constants in $k[x, y, z]$, it may also exists a nonconstant rational function f such that $d(f) = 0$. So the result in [12] mentioned above cannot imply our theorem. Moreover, we give the specific conditions when it has no Darboux polynomials, that is, d has no Darboux polynomials if and only if

$$d = y^{\beta_{12}} z^{\beta_{13}} \frac{\partial}{\partial x} + x^{\beta_{21}} z^{\beta_{23}} \frac{\partial}{\partial y} + x^{\beta_{31}} y^{\beta_{32}} \frac{\partial}{\partial z},$$

where the non-negative integers β_{ij} satisfy neither of the following conditions:

1. $\beta_{12} = \beta_{32}$ or $\beta_{13} = \beta_{23}$ or $\beta_{21} = \beta_{31}$.
2. $\dfrac{\beta_{31}+1}{\beta_{21}+1} = \dfrac{\beta_{12}+1}{\beta_{32}+1} = \dfrac{\beta_{23}+1}{\beta_{13}+1} = 2$ or $\dfrac{\beta_{21}+1}{\beta_{31}+1} = \dfrac{\beta_{32}+1}{\beta_{12}+1} = \dfrac{\beta_{13}+1}{\beta_{23}+1} = 2$.

2 Notations and Preliminaries

In this section, we fix notations and collect some basic facts; see [10] and [13] for details.

A nonzero sequence $\boldsymbol{\gamma} = (\gamma_1, \cdots, \gamma_n)$ of integers is called a direction. For $\boldsymbol{a} = (a_1, \cdots, a_n) \in \mathbf{N}^n$, write $X^{\boldsymbol{a}} = x_1^{a_1} \cdots x_n^{a_n}$ and $\boldsymbol{\gamma a} = \gamma_1 a_1 + \cdots + \gamma_n a_n$. A nonzero polynomial $f \in k[X]$ is said to be $\boldsymbol{\gamma}$-homogeneous of degree s ($s \in \mathbf{Z}$) if $f = \sum\limits_{\boldsymbol{\gamma a} = s} k_{\boldsymbol{a}} X^{\boldsymbol{a}}$, where $k_{\boldsymbol{a}} \in k$. The zero polynomial is seen as a $\boldsymbol{\gamma}$-homogeneous polynomial of an arbitrary degree. A