Communications in Mathematical Research
33(3)(2017), 259–273

Boundedness of Fractional Integrals with a Rough Kernel on the Product Triebel-Lizorkin Spaces

Zhang Hui-hui, Yu Xiao* and Xiang Zhong-qi
(*Department of Mathematics, Shangrao Normal University, Shangrao, Jiangxi, 334001*)

Communicated by Ji You-qing

Abstract: By using the Littlewood-Paley decomposition and the interpolation theory, we prove the boundedness of fractional integral on the product Triebel-Lizorkin spaces with a rough kernel related to the product block spaces.

Key words: fractional integral, block space, Triebel-Lizorkin space, product space

2010 MR subject classification: 42B20, 42B25

Document code: A

Article ID: 1674-5647(2017)03-0259-15

DOI: 10.13447/j.1674-5647.2017.03.06

1 Introduction and Main Results

Let $\mathbf{S}^{N-1}$ be the unit sphere in $\mathbf{R}^N$, $N \geq 2$, with the normalized Lebesgue measure $\mathrm{d}\sigma = \mathrm{d}\sigma(x')$. Define $x' = \dfrac{x}{|x|}$ and $y' = \dfrac{y}{|y|}$. Suppose that a function $\Omega(x', y')$ belongs to $L^1(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$ with $n, m \geq 2$ and satisfies the following two conditions:

$$\Omega(\lambda_1 x,\ \lambda_2 y) = \Omega(x,\ y), \qquad \lambda_1, \lambda_2 \in \mathbf{R}, \tag{1.1}$$

$$\int_{\mathbf{S}^{n-1}} \Omega(x',\ y')\mathrm{d}\sigma(x') = \int_{\mathbf{S}^{m-1}} \Omega(x',\ y')\mathrm{d}\sigma(y') = 0. \tag{1.2}$$

Then the singular integral operator $T_{\Omega,I}$ on the product domain is defined by

$$T_{\Omega,I} f(x,\ y) = p.v. \int_{\mathbf{R}^n \times \mathbf{R}^m} \frac{\Omega(u', v')}{|x|^n |y|^m} f(x-u,\ y-v)\mathrm{d}u\mathrm{d}v. \tag{1.3}$$

For the study of $T_{\Omega,I}$, one may see [1]–[2] for the boundedness of $T_{\Omega,I}$ with $\Omega(x',\ y') \in L^q(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$ or [3]–[5] with $\Omega(x',\ y') \in L(\log^+ L)^2(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$.

Received date: April 4, 2016.
Foundation item: The NSF (11561057, 11226104) of China, the NSF (20151BAB211002, 20151BAB201007) of Jiangxi Province, the Science Foundation (GJJ151054, GJJ151061) of Jiangxi Education Department, and the Scientific Research Project of Shangrao Normal University.
* **Corresponding author**.
E-mail address: zhanghuihuinb@163.com (Zhang H H), yx2000s@163.com (Yu X).

In order to weaken the restriction of $\Omega(x',\ y')$ on $\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}$, in 1995, Jiang and Lu[6] introduced the block function spaces $B_q^{\mu,\nu}(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})$.

Definition 1.1[6] *For $1<q\leq\infty$, a q-block on $\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}$ is an $L^q(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})$ function $b(\ \cdot\ ,\ \cdot\ )$ satisfying*

(i) $\operatorname{supp}(b)\subset Q$, *where Q is an interval on $\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}$, i.e., $Q=Q_1(\xi',\ \delta_1)\times Q_2(\eta',\ \delta_2)$, where*

$$Q_1(\xi',\ \delta_1)=\{x'\in\mathbf{S}^{n-1}:|x'-\xi'|<\delta_1 \text{ for some } \xi'\in\mathbf{S}^{n-1} \text{ and } \delta_1\in(0,1]\},$$
$$Q_2(\eta',\ \delta_2)=\{y'\in\mathbf{S}^{m-1}:|y'-\eta'|<\delta_2 \text{ for some } \eta'\in\mathbf{S}^{m-1} \text{ and } \delta_2\in(0,1]\}.$$

(ii) $\|b\|_{L^q(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})}\leq|Q|^{\frac{1}{q}-1}$, *where $|Q|$ is the volume of Q.*

For $\mu\geq0$ and $\nu\in\mathbf{R}$, a non-negative function $\Phi_{\mu,\nu}$ is defined by

$$\Phi_{\mu,\nu}(t)=\begin{cases}\displaystyle\int_t^1 u^{-1-\mu}\log^{\nu}\frac{1}{u}\mathrm{d}u, & 0<t<1;\\ 0, & t\geq1.\end{cases}$$

Then the definition of the block space $B_q^{\mu,\nu}(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})$ on the product domain is

$$B_q^{\mu,\nu}(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}) =\left\{\Omega\in L^1(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}):\Omega(x',\ y')=\sum_{\ell}C_\ell b_\ell(x',\ y'),\ M_q^{\mu,\nu}(\{C_\ell\})<\infty\right\},\tag{1.4}$$

where each $b_\ell(x',\ y')$ is a q-block supported on Q_ℓ and the definition of $M_q^{\mu,\nu}(\{C_\ell\})$ is defined by

$$M_q^{\mu,\nu}(\{C_\ell\})=\sum_{\ell}|C_\ell|\{1+\Phi_{\mu,\nu}(|Q_\ell|)\}.\tag{1.5}$$

Moreover, the norm of $\Omega\in B_q^{\mu,\nu}(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})$ can be written by

$$N_q^{\mu,\nu}(\Omega)=\inf\left\{\sum_{\ell}|C_\ell|\{1+\Phi_{\mu,\nu}(|Q_\ell|)\}\right\},\tag{1.6}$$

where the infimum is taken over all q-block decompositions of Ω.

Jiang and Lu[6] proved the following theorem.

Theorem 1.1[6] *Suppose that $\Omega\in B_q^{0,\nu}(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})$ with some $q>1$ and $\nu\geq1$. Then the operator $T_{\Omega,I}$ is bounded on $L^2(\mathbf{R}^n\times\mathbf{R}^m)$ for $m\geq2$ and $n\geq2$.*

However, the proof of Theorem 1.1 mainly based on the Plancherel Theorem. By using some basic ideas from [7], Fan *et al.*[8] improved Theorem 1.1 and they proved the following result.

Theorem 1.2[8] *Suppose that $\Omega\in B_q^{0,\nu}(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})$ with some $q>1$ and $\nu\geq1$. Then the operator $T_{\Omega,I}$ is bounded on $L^p(\mathbf{R}^n\times\mathbf{R}^m)$ for $m\geq2$ and $n\geq2$ and $1<p<\infty$.*

On the other hand, the theory of fractional integral operator also plays important roles in harmonic analysis and PDE. Denote $\boldsymbol{\alpha}=(\alpha_1,\ \alpha_2)$ with $0<\alpha_1<n$ and $0<\alpha_2<m$.