COMMUNICATIONS
IN
MATHEMATICAL
RESEARCH
33(4)(2017), 289–303

On the Structure of the Augmentation Quotient Group for Some Non-abelian p-groups

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Communicated by Du Xian-kun

Abstract: In this paper, we study the basis of augmentation ideals and the quotient groups of finite non-abelian p-group which has a cyclic subgroup of index p, where p is an odd prime, and k is greater than or equal to 3. A concrete basis for the augmentation ideal is obtained and then the structure of its quotient groups can be determined.

Key words: integral group ring, augmentation ideal, quotient group, p-group

2010 MR subject classification: 16S34, 20C05

Document code: A

Article ID: 1674-5647(2017)04-0289-15

DOI: 10.13447/j.1674-5647.2017.04.01

1 Introduction

Let G be a finite group, ZG be its integral group ring and the kernel $\Delta(G)$ of the augmentation homomorphism $ZG \to Z$, $\sum\limits_{g\in G} a_g g \to \sum\limits_{g\in G} a_g$, the augmentation ideal of ZG. It is clear that $\Delta(G)$ is the free abelian group on the elements $[g] = g - 1$ for all $g \in G$ modulo the relation $[1] = 0$. The nth power ideal $\Delta^n(G) := (\Delta(G))^n$ of the augmentation ideal $\Delta(G)$ is generated as an abelian group by the products $[g_1, \cdots, g_n] = [g_1] \cdots [g_n]$, $g_1, \cdots, g_n \in G$. It is well known that if G is a finite group of order r, then $\Delta^n(G)$ is a free Z-module of rank $r - 1$ for any $n \geq 1$ (see [1], p.122). The augmentation quotient group is defined as $Q_n(G) = \Delta^n(G)/\Delta^{n+1}(G)$.

Received date: May 6, 2016.
Foundation item: The NSF (11371343, 11461047) of China.
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The problem of determining the structure of augmentation ideals $\Delta^n(G)$ and quotient groups $Q_n(G)$ is an interesting topic in group ring theory. For abelian groups many works have been done (see [1]–[6]). In [2], Hales and Passi (see also [3]) proved that for a finite abelian group G, there exists a number N such that for all $n \geq N$, $Q_n(G)$ is isomorphic to $Q_N(G)$. However, it is usually difficult to write down explicitly a basis of $\Delta^n(G)$ for an arbitrary finite non-abelian group, even for the non-abelian p-group.

For non-abelian finite p-group, if $p = 2$ and every positive integer k is greater than or equal to 4, then there are exactly four isomorphism classes of non-abelian groups of order 2^k which have a cyclic subgroup of index 2. The structure of augmentation quotient groups of all of which are well established, the dihedral group (see [7]), the generalized quaternion group (see [8]), the semidihedral group (or the quasidihedral group) and $M_k(2)$ (see [9]).

If $p \neq 2$ and every positive integer k is greater than or equal to 3, then there is just one isomorphism class of non-abelian groups of order p^k which have a cyclic subgroup of index p. Its presentation is given as follows:

$$\langle a,\, b \mid a^{p^{k-1}} = 1,\ b^p = 1,\ b^{-1}ab = a^{1+p^{k-2}} \rangle.$$

We denote it by M. The current paper investigates the structure of the augmentation ideal and quotient group of the non-abelian p-group M. We prove that for $n \geq N = (p-1)k + 1$,

$$Q_n(M) \cong Q_N(M) \cong Z_{p^{k-2}} \oplus \underbrace{Z_p \oplus \cdots \oplus Z_p}_{(p-1)k+2}.$$

We start with some known results. In [4], Parmenter proved the following theorem.

Theorem 1.1 *Let $G = \langle g \rangle$ be cyclic of order m. Then the set*

$$B_n(G) = \{(g-1)^n,\ (g-1)^{n+1},\ \cdots,\ (g-1)^{n+m-2}\}$$

is a Z-basis for $\Delta^n(G)$.

Let G be a finite group, and denote by $G_1 = [G, G]$ the commutator subgroup of G. For $i \geq 1$, define $G_i = [G, G_{i-1}]$. Then we have the sequence: $G = G_0 \triangleright G_1 \triangleright G_2 \triangleright \cdots$. In [8], Zhou and You gave the following theorem.

Theorem 1.2 $g - 1 \in \Delta^{i+1}(G)$, *if $g \in G_i$.*

2 Structure of $Q_n(M)$ for the Non-abelian p-group M

Let

$$M = \langle a,\, b \mid a^{p^{k-1}} = 1,\, b^p = 1,\, b^{-1}ab = a^{1+p^{k-2}} \rangle$$

be a finite non-abelian p-group of order p^k which have a cyclic subgroup of index p, where $p \neq 2$, $k \geq 3$. It is not hard to see that

$$M = \{b^t a^u \mid 0 \leq t \leq p-1,\ 0 \leq u \leq p^{k-1} - 1\},$$

and

$$b^t a^u \cdot b^i a^j = b^{t+i} a^{u(1+p^{k-2})^i + j}.$$

Consequently, we have