Communications
in
Mathematical
Research
33(4)(2017), 311–317

A Note on a Functional Differential Equation with State Dependent Argument

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Communicated by Li Yong

Abstract: This paper is concerned with solutions of a functional differential equation. Using Krasnoselskii's fixed point theorem, the solutions can be obtained from periodic solutions of a companion equation.

Key words: iterative functional differential equation, periodic solution, fixed point theorem

2010 MR subject classification: 39B12, 39B82

Document code: A

Article ID: 1674-5647(2017)04-0311-07

DOI: 10.13447/j.1674-5647.2017.04.03

1 Introduction

Recently, iterative functional differential equations of the form

$$x'(t) = x(t - \sigma(t))$$

has appeared in several papers. However, such equations, about the periodic solution of the delay function $\sigma(t)$ depends on the argument of the unknown function have been relatively little researched. McKiernan[1] considered analytic solutions of the problem

$$x'(t) = \frac{1}{x(x(t))}, \qquad x(\mu) = \mu.$$

In [2]–[4], the authors studied the equations

$$x'(t) = x(at + bx(t)), \tag{1.1}$$

$$x'(t) = \frac{1}{x(at + bx(t))}$$

and

$$\alpha t + \beta x'(t) = x(at + bx'(t))$$

Received date: June 7, 2016.

Foundation item: The NSF (11326120 and 11501069) of China, the Foundation (KJ1400528 and KJ1600320) of Chongqing Municipal Education Commission and the Foundation (02030307-00039) of Youth Talent of Chongqing Normal University.

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established sufficient conditions for the existence of analytic solutions in some local neighbourhoods of the fixed points of the functions.

Since Burton[5] applied Krasnoselskii's fixed theorem to prove the existence of periodic solutions, and this theorem has been extensively used in proving stability, periodic of solutions and boundedness of solutions in functional differential (difference) equations. 2005, Raffoul[6] used fixed point theorem to show a nonlinear neutral system

$$\frac{\mathrm{d}}{\mathrm{dt}}[x(t) - ax(t-\tau)] = r(t)x(t) - f(t, x(t-\tau))$$

has a periodic solution. In [7], Guo and Yu discussed the existence and multiplicity of periodic of the second order difference equation. Some other works can also be found in [8]–[12].

In this note, we show the solutions of equation (1.1) have a relation with a periodic function. In fact, letting

$$y(t) = at + bx(t), \qquad b \neq 0,$$

(1.1) can be changed into

$$y'(t) = a + y(y(t)) - ay(t). \tag{1.2}$$

Under certain conditions, we can find the equation (1.2) has periodic solutions $y(t)$, and the solutions for the original equation can be obtained.

For convenience, we make use $C(\mathbf{R}, \mathbf{R})$ to denote the set of all real valued continuous functions from $\mathbf{R}$ into $\mathbf{R}$. For $T > 0$, define

$$\mathcal{P}_T = \Big\{ f \in C(\mathbf{R}, \mathbf{R}) : f(t+T) = f(t),\ \forall t \in \mathbf{R} \Big\}.$$

Then $\mathcal{P}_T$ is a Banach space with the norm

$$\|x\| = \max_{t\in[0,T]} |x(t)| = \max_{t\in\mathbf{R}} |x(t)|.$$

For P, $L \geq 0$, define the set

$$\mathcal{P}_T(P, L) = \Big\{ f \in \mathcal{P}_T : \|f\| \leq P,\ |f(t_2) - f(t_1)| \leq L|t_2 - t_1|,\ \forall t_1, t_2 \in \mathbf{R} \Big\},$$

which is a closed convex and bounded subset of $\mathcal{P}_T$, and we wish to find T-periodic functions $y \in \mathcal{P}_T(P, L)$ satisfies (1.2).

2 Periodic Solutions of (1.2)

In this section, the existence of periodic solutions of the equation (1.2) is proved. Now let us state the Krasnoselskii's fixed point theorem, it is used to prove our main theorem.

Theorem 2.1 [13] *Let Ω be a closed convex nonempty subset of a Banach space $(\mathbb{B}, \|\cdot\|)$. Suppose that A and B map Ω into $\mathbb{B}$ such that*

(i) *A is compact and continuous*;

(ii) *B is a contraction mapping*;

(iii) *$x, y \in \Omega$ implies $Ax + By \in \Omega$.*

Then there exists a $z \in \Omega$ with $z = Az + Bz$.

We begin with the following lemma.