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A New Hybrid Algorithm and Its Numerical Realization for a Quasi-nonexpansive Mapping

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Abstract: The purpose of this article is to propose a new hybrid projection method for a quasi-nonexpansive mapping. The strong convergence of the algorithm is proved in real Hilbert spaces. A numerical experiment is also included to explain the effectiveness of the proposed methods. The results of this paper are interesting extensions of those known results.
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1 Introduction

Suppose that H is a real Hilbert space. We denote by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ the inner product and the norm, respectively. Suppose that C is a closed convex nonempty subset of H. We denote by $F(T)$ the fixed point set of a mapping $T\colon C \to C$, i.e., $F(T) = \{x \in C\colon x = Tx\}$. A mapping $T\colon C \to C$ is called a nonexpansive mapping if

$$\|Tx - Ty\| \leq \|x - y\|, \qquad x, y \in C.$$

A mapping $T\colon C \to C$ is called a quasi-nonexpansive mapping if $F(T) \neq \emptyset$ such that

$$\|Tx - p\| \leq \|x - p\|, \qquad x \in C,\ p \in F(T).$$

Obviously, a nonexpansive mapping with a nonempty fixed point set $F(T)$ is a quasi-nonexpansive mapping, but the converse may be not true.

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The construction of fixed points for nonlinear mappings is of practical importance. In particular, iterative algorithms for finding fixed points of nonexpansive mappings have received extensive investigation (see [1]–[2]) since these algorithms have a variety of applications in inverse problem, image recovery and signal processing (see [3]–[5]).

Nakajo and Takahashi[6] first introduced a hybrid algorithm for a nonexpansive mapping. Thereafter, some hybrid algorithms have been studied extensively since they have strong convergence (see [7]–[13]). Recently, Dong and Lu[14] proposed the following hybrid iterative method for a nonexpansive mapping in a Hilbert space H and gave numerical examples to describe the effectiveness of the new algorithm:

$$\begin{cases} x_0,\ z_0 \in C \text{ chosen arbitrarily}, \\ z_{n+1} = \alpha_n z_n + (1-\alpha_n)Tx_n, \\ C_n = \{z \in C\colon \|z_{n+1} - z\|^2 \le \alpha_n \|z_n - z\|^2 + (1-\alpha_n)\|x_n - z\|^2\}, \\ Q_n = \{z \in C\colon \langle x_n - z,\, x_n - x_0\rangle \le 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0). \end{cases} \tag{1.1}$$

Dong and Lu[14] showed the following theorem.

Theorem 1.1[14] *Let C be a closed convex subset of a Hilbert space H, and let $T\colon C \to C$ be a nonexpansive mapping such that $F(T) \ne \emptyset$. Assume that $\{\alpha_n\} \subset [0,\, \sigma]$ holds for some $\sigma \in \left[0,\, \frac{1}{2}\right)$. Then $\{x_n\}$ and $\{z_n\}$ generated by algorithm (1.1) converge strongly to $P_{F(T)}x_0$.*

On the basis of [14], we design a simple hybrid method for a quasi-nonexpansive mapping. A strong convergence theorem is proved by using new methods in this paper. We also give a numerical experiment to describe the effectiveness of the proposed algorithm. The results of this paper improve the related ones obtained by some authors (e.g., [6] and [14], etc.).

2 Preliminaries

Lemma 2.1[8] *Let K be a closed convex subset of real Hilbert space H. Given $x \in H$ and $z \in K$. Then $z = P_K x$ if and only if there holds the relation*

$$\langle x - z,\, y - z\rangle \le 0, \qquad y \in K.$$

Lemma 2.2[14] *Suppose that $\{a_n\}$ and $\{b_n\}$ are nonnegative real sequences, $\alpha \in [0,\, 1)$, $\beta \in \mathbf{R}^+$, and for any $n \in \mathbf{N}$, the following inequality holds:*

$$a_{n+1} \le \alpha a_n + \beta b_n.$$

If $\sum\limits_{n=1}^{\infty} b_n < +\infty$, then $\lim\limits_{n\to\infty} a_n = 0$.

Lemma 2.3 *Let C be a closed convex nonempty subset of H, $T\colon C \to C$ be a quasi-nonexpansive mapping. Then $F(T)$ is a convex closed subset of C.*

Proof. We prove first that $F(T)$ is closed. Let $\{p_n\} \subset F(T)$ with $p_n \to p$ $(n \to \infty)$. We prove that $p \in F(T)$. Since T is quasi-nonexpansive, one has

$$\|Tp - p_n\| \le \|p - p_n\|,$$