Communications
in
Mathematical
Research
34(1)(2018), 36–46

Reversible Properties of Monoid Crossed Products

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Communicated by Du Xian-kun

Abstract: We study the reversible properties of monoid crossed products. The new class of strongly CM-reversible rings is introduced and characterized. This class of rings is a generalization of those of strongly reversible rings, skew strongly reversible rings and strongly M-reversible rings. Some well-known results on this subject are generalized and extended.

Key words: monoid crossed product, strongly reversible ring, strongly CM-reversible ring

2010 MR subject classification: 16S35, 16U20, 13B02

Document code: A

Article ID: 1674-5647(2018)01-0036-11

DOI: 10.13447/j.1674-5647.2018.01.04

1 Introduction

Throughout, unless otherwise indicated, R denotes an associative ring with identity and M is a monoid. In [1], Cohn introduced the notion of a reversible ring. A ring R is said to be reversible if $ab = 0$ implies $ba = 0$ for all $a, b \in R$. Anderson and Camillo[2] used the term of ZC_2 for what is called reversible. It was proved in [3] that polynomial rings over reversible rings need not be reversible. A ring R is called reduced if it has no non-zero nilpotent elements (see [4]), i.e., $a^2 = 0$ implies $a = 0$ for all $a \in R$. Recall from [5] that a ring R is strongly reversible if polynomials $f(x)$, $g(x) \in R[x]$ with $f(x)g(x) = 0$ implies $g(x)f(x) = 0$. It is clear that all reduced rings are strongly reversible, but the inverse is not true. Rage and Chhawchharia[6] introduced the concept of an Armendariz ring. A

Received date: Sept. 9, 2016.

Foundation item: The NSF (11601005) of China, the Jiangsu Planned Projects (1601151C) for Postdoctoral Research Funds, the Provincial NSF (KJ2017A040) of Anhui Province, and the Graduate Students Innovation Projects (2016141) of Anhui University of Technology.

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ring R is an Armendariz ring, whenever polynomials $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$, $g(x) = b_0 + b_1x + b_2x^2 + \cdots + b_mx^m$ are in $R[x]$ and if $f(x)g(x) = 0$, then $a_ib_j = 0$ for all i, j. In the following, we denote by $R[M]$ the monoid ring constructed from ring R and the monoid M, and e always stands for the identity of M. According to [7], a ring R is called an M-Armendariz if $\alpha = a_1g_1 + a_2g_2 + \cdots + a_ng_n$, $\beta = b_1h_1 + b_2h_2 + \cdots + b_mg_m \in R[M]$ satisfy $\alpha\beta = 0$, then $a_ib_j = 0$ for all i, j. A ring R is strongly M-reversible if $\alpha\beta = 0$ implies $\beta\alpha = 0$ for all $\alpha, \beta \in R[M]$ (see [8]). Recall from [9] that a ring R is skew strongly M-reversible whenever $\alpha\beta = 0$ implies $\beta\alpha = 0$, where $\alpha, \beta \in R * M$.

A monoid M is a u.p.-monoid (unique product monoid) if for any two nonempty finite subsets $A, B \in M$ there exists an element $g \in M$ uniquely in the form of ab with $a \in A$ and $b \in B$. If there exists a monoid homomorphism $\omega : M \to \mathrm{Aut}(R)$, we denote by $\omega_g(r)$ the image of r under $\omega(g)$ with $g \in M$ and $r \in R$. We can form a skew monoid ring $R * M$ (see [10]) (induced by the monoid homomorphism ω) by taking its elements to be finite formal combinations $\sum\limits_{i=1}^{n} a_ig_i$ with the multiplication induced by $(ag)(bh) = (a\omega_g(b))(gh)$. The map $\omega : M \to \mathrm{Aut}(R)$ defined by $\omega_g(r) = r$ for each $g \in M$ and $r \in R$ is called the trivial monoid homomorphism. More generally, if R is a ring and M is a monoid, then the crossed product $R\sharp M$ over R consists of all finite sums $R\sharp M = \{\sum r_gg \mid r_g \in R,\ g \in M\}$ with addition defined componentwise and multiplication defined by the distributive law and two rules that are called the twisting and the action explained below. Specifically, we have the twisting operation $gh = f(g, h)gh$ for every $g, h \in M$, where $f : M \times M \to U = U(R)$. For every $r \in R$ and $g \in M$, we have $gr = \omega_g(r)g$ with $\omega : M \to \mathrm{Aut}(R)$. If $R\sharp M$ is the crossed product over R, then the twisted function f and the weak action ω of M on R must satisfy

$$\begin{aligned}
&\omega_g(\omega_h(r)) = f(g,\, h)\omega_{gh}(r)f(g,\, h)^{-1},\\
&\omega_g(f(h,\, k))f(g,\, hk) = f(g,\, h)f(gh,\, k),\\
&f(e,\, g) = f(g,\, e) = 1
\end{aligned}$$

for all $g,\ h,\ k \in M$.

Monoid crossed products are a quite general ring construction. Let $R\sharp M$ be a monoid crossed product with twisting f and action ω. If the twisting f is trivial, i.e., $f(x, y) = 1$ for all $x,\ y \in M$, then $R\sharp M$ is the skew monoid ring $R * M$. If the action ω is trivial, i.e., $\omega_g = i_R$ with i_R the identity map over R, then $R\sharp M$ is the twisted monoid ring $R^\tau[M]$. If both the twisting f and the action ω are trivial, then $R\sharp M$ is a monoid ring, denoted by $R[M]$. Motivated by the results of [3], [5], [8] and [9], in this paper we introduce and study the concept of strongly CM-reversible rings, which is a generalization of strongly reversible rings, strongly M-reversible rings and skew strongly M-reversible rings. The main idea is to study the reversible condition defined for the monoid ring crossed product $R\sharp M$. It is shown that if R is an M-rigid ring, then R is strongly CM-reversible. Moreover, if R is a right Ore ring with classical right quotient ring Q, then we show that R is strongly CM-reversible if and only if Q is strongly CM-reversible. Suppose that R/I is strongly CM-reversible for some ω-invariant ideal I of R. If I is an M-rigid ring, it is proved that R is strongly