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On the Coefficients of Several Classes of Bi-univalent Functions Defined by Convolution

Guo Dong[1] and Li Zong-tao[2]
(*1. Foundation Department, Chuzhou Vocational and Technical College, Chuzhou, Anhui, 239000*)
(*2. Foundation Department, Guangzhou Civil Aviation College, Guangzhou, 510403*)

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Abstract: In this paper, we introduce several new subclasses of the function class Σ of bi-univalent functions analytic in the open unit disc defined by convolution. Furthermore, we investigate the bounds of the coefficients $|a_2|$ and $|a_3|$ for functions in these new subclasses. The results presented in this paper improve or generalize the recent works of other authors.
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1 Introduction

Let A denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{+\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $U = \{z\colon |z| < 1\}$. Further, we denote by S the class of all functions in A which are univalent in U. A function f in S is said to be starlike of order α, $0 \le \alpha < 1$, and is denoted by $S^*(\alpha)$ if $\mathrm{Re}\left\{\dfrac{zf'(z)}{f(z)}\right\} > \alpha$, $z \in U$, and is said to be convex of order α, $0 \le \alpha < 1$, and is denoted by $K(\alpha)$ if $\mathrm{Re}\left\{1 + \dfrac{zf''(z)}{f'(z)}\right\} > \alpha$,

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E-mail address: gd791217@163.com (Guo D).

$z \in U$. Mocanu[1] studied linear combinations of the representations of convex and starlike functions and defined the class of α-convex functions. In [2], it was shown that if

$$\mathrm{Re}\left\{(1-\alpha)\frac{zf'(z)}{f(z)}+\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)\right\}>0, \qquad z \in U,$$

then f is in the class of starlike functions $S^*(0)$ for α be a real number and is in the class of convex functions $K(0)$ for $\alpha \geq 1$.

Further, We say that $f(z) \in A$ is α-starlike in U if $f(z)$ satisfies

$$f(z)f'(z)\frac{1+zf''(z)}{f'(z)} \neq 0, \qquad |z|<1$$

and

$$\mathrm{Re}\left\{\left(\frac{zf'(z)}{f(z)}\right)^{\alpha}\left(1+\frac{zf''(z)}{f'(z)}\right)^{1-\alpha}>0\right\}.$$

For such α-starlike functions, Lewandowski *et al.*[3] proved that all α-starlike functions are univalent and starlike for all α ($\alpha \in \mathbf{R}$).

In [4], it was shown that if

$$\mathrm{Re}\left(\frac{\alpha z^2 f''(z)}{f(z)}+\frac{zf'(z)}{f(z)}\right)>-\frac{\alpha}{2}, \qquad \alpha \geq 0,\ z \in U,$$

then $f \in S^*(0)$.

For the function $f(z)=z+\sum\limits_{n=2}^{+\infty} a_n z^n$ and $g(z)=z+\sum\limits_{n=2}^{+\infty} b_n z^n$, let $(f*g)(z)$ denote the Hadamard product or convolution of $f(z)$ and $g(z)$, defined by

$$(f*g)(z)=z+\sum_{n=2}^{+\infty} a_n b_n z^n. \tag{1.2}$$

For $0 \leq \alpha < 1$ and $\lambda \geq 0$, we let $Q_\lambda(h, \alpha)$ be the subclass of A consisting of functions $f(z)$ of the form (1.1) and functions $h(z)$ given by

$$h(z)=z+\sum_{n=2}^{+\infty} h_n z^n, \qquad h_n>0 \tag{1.3}$$

and satisfying the analytic criterion:

$$\mathrm{Re}\left[(1-\lambda)\frac{(f*h)(z)}{z}+\lambda(f*h)'(z)\right]>\alpha, \qquad 0 \leq \alpha < 1,\ \lambda \geq 0.$$

It is easy to see that $Q_{\lambda_1}(h, \alpha) \subset Q_{\lambda_2}(h, \alpha)$ for $\lambda_1 > \lambda_2 \geq 0$. Thus, for $\lambda \geq 1$, $0 \leq \alpha < 1$, $Q_\lambda(h,\ \alpha) \subset Q_1(h,\ \alpha)=\{f, h \in A\colon \mathrm{Re}(f*h)'(z)>\alpha,\ 0 \leq \alpha < 1\}$ and hence $Q_\lambda(h,\ \alpha)$ is univalent class (see [5]–[7]).

We note that $Q_\lambda\left(\dfrac{z}{1-z},\ \alpha\right)=Q_\lambda(\alpha)$ (see [8]).

It is well known that every function $f \in S$ has an inverse f^{-1}, defined by

$$f^{-1}(f(z))=z, \qquad z \in U$$

and

$$f(f^{-1}(\omega))=\omega, \qquad |\omega|<r_0(f),\ r_0(f) \geq \frac{1}{4},$$