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Some Normality Criteria for Families of Meromorphic Functions

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Abstract: Let k be a positive integer and $\mathcal{F}$ be a family of meromorphic functions in a domain D such that for each $f \in \mathcal{F}$, all poles of f are of multiplicity at least 2, and all zeros of f are of multiplicity at least $k+1$. Let a and b be two distinct finite complex numbers. If for each $f \in \mathcal{F}$, all zeros of $f^{(k)} - a$ are of multiplicity at least 2, and for each pair of functions f, $g \in \mathcal{F}$, $f^{(k)}$ and $g^{(k)}$ share b in D, then $\mathcal{F}$ is normal in D.

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1 Introduction and Main Results

First of all we recall that a family $\mathcal{F}$ of functions meromorphic in a plane domain D is called to be normal in D, in the sense of Montel, if every sequence $\{f_n\} \subset \mathcal{F}$ contains a subsequence $\{f_{n_j}\}$ which converges spherically locally uniformly in D, to a meromorphic function or the constant ∞ (see [1]–[3]).

Let f and g be meromorphic in a domain D, $b \in \mathbf{C} \bigcup \{\infty\}$. If $f(z) - b$ and $g(z) - b$ assume the same zeros ignoring multiplicity, we say that f and g share b in D.

Inspired by heuristic Bloch's principle (see [4]–[5]) that there is an analogue in normal family theory corresponding to every Liouville-Picard type theorem, Gu[6] proved the following famous normality criterion related to the well-known Hayman's alternative (see [7]).

Theorem A[6] *Let $\mathcal{F}$ be a family of meromorphic functions in a domain D, k be a positive integer, and b be a nonzero finite complex number. If for each $f \in \mathcal{F}$, $f \neq 0$ and $f^{(k)} \neq b$ in D, then $\mathcal{F}$ is normal in D.*

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Recently, by the idea of shared values, Fang and Zalcman[8],[9] extended Theorem A as follows.

Theorem B[8],[9] *Let k be a positive integer and $\mathcal{F}$ be a family of meromorphic functions in a domain D such that for each $f \in \mathcal{F}$, all zeros of f are of multiplicity at least $k+2$. Let a and $b \neq 0$ be two finite complex numbers. If for each pair of functions f, $g \in \mathcal{F}$, f and g share a, $f^{(k)}$ and $g^{(k)}$ share b in D, then $\mathcal{F}$ is normal in D.*

In 1989, Schwick[10] obtained the following theorem.

Theorem C[10] *Let $\mathcal{F}$ be a family of meromorphic functions in a domain D, n, k be positive integers with $n \geq k+3$, and b be a nonzero finite complex number. If for each $f \in \mathcal{F}$, $(f^n)^{(k)} \neq b$ in D, then $\mathcal{F}$ is normal in D.*

In 2009, Li and Gu[11] improved Theorem C and proved the following result with the idea of shared values.

Theorem D[11] *Let $\mathcal{F}$ be a family of meromorphic functions in a domain D, n, k be positive integers with $n \geq k+2$, and b be a nonzero finite complex number. If for each pair of functions f, $g \in \mathcal{F}$, $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share b in D, then $\mathcal{F}$ is normal in D.*

In 1998, Wang and Fang[12] proved the following theorem.

Theorem E[12] *Let k be a positive integer and $\mathcal{F}$ be a family of meromorphic functions in a domain D such that for each $f \in \mathcal{F}$, all poles of f are of multiplicity at least 2, and all zeros of f are of multiplicity at least $k+1$. Let b be a nonzero finite complex number. If for each $f \in \mathcal{F}$, $f^{(k)} \neq b$ in D, then $\mathcal{F}$ is normal in D.*

It is natural to ask whether Theorem E can be extended in the same way that Theorem B extends Theorem A or Theorem D extends Theorem C. In this paper, we offer such an extension.

Theorem 1.1 *Let k be a positive integer and $\mathcal{F}$ be a family of meromorphic functions in a domain D such that for each $f \in \mathcal{F}$, all poles of f are of multiplicity at least 2, and all zeros of f are of multiplicity at least $k+1$. Let a and b be two distinct finite complex numbers. If for each $f \in \mathcal{F}$, all zeros of $f^{(k)} - a$ are of multiplicity at least 2, and for each pair of functions f, $g \in \mathcal{F}$, $f^{(k)}$ and $g^{(k)}$ share b in D, then $\mathcal{F}$ is normal in D.*

Corollary 1.1 *Let k be a positive integer and $\mathcal{F}$ be a family of holomorphic functions in a domain D such that for each $f \in \mathcal{F}$, all zeros of f are of multiplicity at least $k+1$. Let a and b be two distinct finite complex numbers. If for each $f \in \mathcal{F}$, all zeros of $f^{(k)} - a$ are of multiplicity at least 2, and for each pair of functions f, $g \in \mathcal{F}$, $f^{(k)}$ and $g^{(k)}$ share b in D, then $\mathcal{F}$ is normal in D.*

Moreover, we can prove the following result by restricting the numbers of the zeros of $f^{(k)} - b$.