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Ulam-Hyers Stability of Trigonometric Functional Equation with Involution

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Abstract: The present work aims to determine the solution of trigonometric functional equation f with involution from group to field by using the properties of involution function, and the solution and Ulam-Hyers stability of the trigonometric functional equation are also discussed. Furthermore, this method generalizes the main theorem and gives the supplement in some reference.

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1 Introduction

Let G be an abelian group, $\mathbf{R}, \mathbf{C}$ the set of real numbers, complex numbers, respectively. Hom$(G,\ F)$ denotes the group of homomorphisms from the group G to the multiplicative field F. Every element in Hom$(G,\ F)$ is called a (group) character and denoted by χ. Thus a character is a nonzero multiplicative homomorphism from group G into the multiplicative group of nonzero complex numbers. If χ is a group character of G, then we denote $\chi(x^{-1})$ by $\check{\chi}(x)$. Clearly, $\check{\chi}(x)$ is also a character of G. Denote $\sigma\colon G \to G$ be an involution, provided that $\sigma(x+y) = \sigma(x) + \sigma(y)$ and $\sigma(\sigma(x)) = x$ for all $x,\ y \in G$. For convenience we always denote $\sigma(x)$ as simply σx. A function $m\colon G \to F$ is called an exponential function provided that $m(x+y) = m(x)m(y)$ for all $x,\ y \in G$. A function $f\colon G \to F$ is said to be an abelian function if and only if $f(xy) = f(yx)$ for all $x, y \in G$. And a function $f\colon G \to F$ is said to be σ-odd with respect to an involution $\sigma\colon G \to G$ if and only if $f(\sigma x) = -f(x)$ for all $x \in G$. It is easy to see that if f is a σ-odd function, then $f(e) = 0$, where e is a unit element in G. Similarly, a function $f\colon G \to F$ is said to be σ-even with respect to an involution $\sigma\colon G \to G$

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if and only if $f(\sigma x) = f(x)$ for all $x \in G$. A function $m\colon G \to F$ is called σ-exponential if m satisfies $m(xy) = m(x)m(y)$ and $m(\sigma x) = m(x)$ for $x \in G$, and denoted by m_σ.

In 1910, Vleck[1] considered the functional equation

$$f(x-y+a) - f(x+y+a) = 2f(x)f(y), \qquad x, y \in G, \tag{1.1}$$

where $a > 0$ is fixed. He proved that f is a periodic function with period $4a$ and (1.1) implies the cosine functional equation.

In 2009, Kannappan[2] considered the functional equation

$$f(x-y+a) + f(x+y+a) = 2f(x)f(y), \qquad a > 0,\ x, y \in G, \tag{1.2}$$

and proved the following result: the general solution $f\colon \mathbf{R} \to \mathbf{C}$ of (1.2) is either $f = 0$ or $f(x) = g(x-a)$, where g is an arbitrary solution of the cosine functional equation

$$g(x+y) + g(x-y) = 2g(x)g(y), \qquad x,\ y \in \mathbf{R}$$

with period $2a$.

In 2002, Czerwi[3] studied the functional equation

$$f(x+y+a) + f(y-x-a) = 2f(x)f(y), \qquad a > 0,\ x, y \in G \tag{1.3}$$

on a locally compact abelian group G. And this work was extended by Fechner[4] in 2009.

Serval years later, Perkins and Sahoo[5] generalized the above functional equation

$$f(x-y+a) + g(x+y+a) = 2f(x)f(y), \qquad x,\ y \in G \tag{1.4}$$

Generally, these above functional equations are called trigonometric functions. More stability and solution of trigonometric functions which was solved in the reference [6]–[12].

On the other hand, Hyers[10] proposed the Hyers-Ulam stability problems of functional equations which were concerning the approximate homomorphisms from a group to a metric group. Later, Sźekelyhidi[11] investigated the Hyers-Ulam stability question of the following trigonometric functional equations

$$f(x+y) = f(x)g(y) + g(x)f(y), \qquad x,\ y \in G, \tag{1.5}$$

$$f(x+y) = f(x)g(y) - f(x)f(y), \qquad x,\ y \in G. \tag{1.6}$$

As a particular case of the result in [11], he obtained the stability of the functional inequalities

$$|f(x+y) - f(x)g(y) - g(x)f(y)| \le \varphi(y), \tag{1.7}$$

$$|f(x+y) - f(x)g(y) + g(x)f(y)| \le \varphi(y) \tag{1.8}$$

for all $x, y \in G$, where $f\colon G \to \mathbf{C}$ and $\varphi\colon G \to \mathbf{R}^+$.

From above reference, we found that though some researchers investigated solutions and stability of functional equations with sine functions and cosine functions (see [10]–[11]), there were some pity needing to be completed. Secondly, there were similar ways to obtain their solutions.

Based on this, we determine the general solution of the following functional equations (1.9) and (1.10) with involution on groups.

$$f(x\sigma y) = f(x)f(y) + g(x)g(y), \tag{1.9}$$

$$f(x\sigma y) = f(x)f(y) - g(x)g(y). \tag{1.10}$$