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# Spanning Pre-disks in a Compression Body

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**Abstract:** A properly embedded essential planar surface $P$ (not a disk) in a compression body $V$ is called a spanning pre-disk with respect to $J$, if one boundary component of $P$ is lying in $\partial_+ V$ and all other boundary components of $P$ are lying in $\partial_- V$ and coplanar with $J$. In this paper, we show that the number of boundary components of spanning pre-disks in a compression body is unbounded. But the number of a maximal collection of spanning pre-disks is bounded.

**Key words:** spanning pre-disk, curve complex, compression body, maximal collection

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## 1 Introduction

Let $M$ be an orientable compact 3-manifold. A natural question is whether there exists a properly embedded connected incompressible surface in $M$ with genus $g$ and $b$ boundary components for given $g$ and $b$. Jaco[1] showed that the answer is positive when $b$ equals to 1 or 2 for the handlebody of genus 2 (therefore, for the handlebody of genus $n \geq 2$). The examples constructed by Jaco are non-separating in the handlebody. Examples of such separating surfaces in a handlebody were given independently by Eudave-Muñoz[2], Howards[3] and Qiu[4]. Nogueira and Segerman[5] gave a generalized description of such surfaces in a handlebody with genus at least 2 or a 3-manifold with a compressible boundary component with genus at least 2.

Another question is whether the number of components in a maximal collection of pair-

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wise disjoint, non-parallel, incompressible surfaces in a compact 3-manifold is bounded. The Kneser-Haken Finiteness Theorem says that this is true if the surfaces are further assumed to be $\partial$-incompressible (for a proof see [1] and [6]). The conclusion is not true if the assumption of the $\partial$-incompressibility for the surfaces is removed. On the other hand, B. Freedman and M. H. Freedman[7] showed that for a given compact 3-manifold if the Betti numbers of surfaces are bounded, then the number of surfaces is bounded. Eudave-Muñoz and Shor[8] showed that there is a bound of the number of surfaces depending on the Heegaard genus of 3-manifold and the Betti numbers of surfaces. There are also other results about the embedding of a maximal collection of essential annuli in a handlebody, see [9]–[11].

The pre-disk in a 3-manifold was first introduced by Jaco[12]. Let $M$ be a 3-manifold, and $J$ an essential simple closed curve on a boundary component $F$. An essential planar surface $P$ properly embedded in $M$ is called a pre-disk with respect to $J$ if one boundary component $C$ of $P$ is not coplanar with $J$, and all other boundary components of $P$ are coplanar with $J$ in $F$. Jaco showed if $\partial M - J$ is incompressible, then there is no properly embedded pre-disk with respect to $J$ in $M$. A handle addition theorem was given by Jaco as an application of this result.

We consider spanning pre-disks in a compression body. Let $V$ be a nontrivial compression body with $\partial_- V \neq \emptyset$ and $J$ an essential simple closed curve in $\partial_- V$. A properly embedded essential planar surface $P$ (not a disk) in $V$ is called a spanning pre-disk with respect to $J$, if one boundary component of $P$ is lying in $\partial_+ V$ and all other boundary components of $P$ are lying in $\partial_- V$ and coplanar with $J$.

Let $V$ be a nontrivial compression body and $F$ a component of $\partial_- V$. Then we have the following theorem:

**Theorem 1.1** *Let $C$ be an essential simple closed curve in $\partial_- V$ and $n$ a positive integer. If there exists a non-separating essential disk in $V$ or the component of $\partial_- V$ containing $C$ has genus at least 2, then there is a spanning pre-disk $P$ with respect to $C$ in $V$ such that $|\partial P| \geq n$.*

Let $\mathscr{C}$ be a collection of mutually disjoint spanning pre-disks with respect to $C$ in $V$. $\mathscr{C}$ is called to be maximal if whenever $P$ is a spanning pre-disk with respect to $C$ with $P \cap \mathscr{C} = \emptyset$, then $P$ is parallel to a component of $\mathscr{C}$ in $V$. Then we have the following theorem:

**Theorem 1.2** *Let $V$ be a nontrivial compression body with $\partial_- V \neq \emptyset$ and $C$ an essential simple closed curve in $\partial_- V$. If the collection $\mathscr{C}$ is maximal, then*

$$|\mathscr{C}| \leq 3g(\partial_+ V) - 3.$$

The article is organized as follows. In Section 2, we review some necessary preliminaries. A key lemma is given in Section 3. The proofs of the main results are given in Section 4.

## 2 Preliminaries

Let $V$ be a nontrivial compression body. A set $\mathcal{D}$ of disjoint essential disks in $V$ is called to be a minimal complete collection if $V - \mathcal{D}$ is homeomorphic to $\partial_- V \times I$. Assume that