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Fekete-Szegö Problem for a Subclass of Meromorphic Functions Defined by the Dziok-Srivastava Operator

Guo Dong[1], Li Zong-tao[2] and Xiong Liang-peng[3]

(*1. Foundation Department, Chuzhou Vocational and Technical College, Chuzhou, Anhui, 239000*)

(*2. Foundation Department, Guangzhou Civil Aviation College, Guangzhou, 510403*)

(*3. School of Mathematics and Statistics, Wuhan University, Wuhan, 430072*)

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Abstract: By using the hypergeometric function defined by the Dziok-Srivastava operator, a new subclass of meromorphic function is introdued. We obtain Fekete-Szegö inequalities for the meromorphic function $f(z)$ for which $\alpha - \dfrac{1+\alpha\left\{1+\dfrac{z[{}_lI_m f(z)]''}{[{}_lI_m f(z)]'}\right\}}{\dfrac{z[{}_lI_m f(z)]'}{{}_lI_m f(z)}}$ $\prec \varphi(z)$ $\left(\alpha \in \mathbf{C} - \left\{\dfrac{1}{2}, 1\right\}\right)$.

Key words: analytic and meromorphic function, starlike and convex function, hypergeometric function, Fekete-Szegö problem, Dziok-Srivastava operator, Hadamard product

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1 Introduction and Definition

Let Σ denote the class of meromorphic functions of the form:

$$f(z) = \frac{1}{z} + \sum_{n=0}^{+\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk

$$U^* = \{z\colon z \in \mathbf{C},\ 0 < |z| < 1\} = U - \{0\}.$$

A function $f \in \Sigma$ is meromorphic starlike of order β, denoted by $S^*(\beta)$, if $\mathrm{Re}\left\{\dfrac{zf'(z)}{f(z)}\right\} < -\beta$

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E-mail address: gd791217@163.com (Guo D).

$(0 \leq \beta < 1,\ z \in U^*)$. A function $f \in \Sigma$ is meromorphic convex of order β, denoted by $C^*(\beta)$, if $\mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} < -\beta\ (0 \leq \beta < 1,\ z \in U^*)$.

Let φ be an analytic function with positive real part in the open unit disk U, $\varphi(0) = 1$, $\varphi'(0) > 0$ and $\varphi(U)$ be symmetric with respect to the real axis. The Taylor's series expansion of such function is of the form

$$\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots \tag{1.2}$$

Aouf[1] introduced and studied the class $\mathcal{F}_\alpha^*(\varphi)$, which consists of functions $f(z) \in \Sigma$ for

$$-\frac{zf'(z) + \alpha z^2 f''(z)}{(1-\alpha)f(z) + \alpha z f'(z)} \prec \varphi(z), \qquad \alpha \in \mathbf{C} - (0,\ 1].$$

For the functions

$$f(z) = \frac{1}{z} + \sum_{n=0}^{+\infty} a_n z^n, \qquad g(z) = \frac{1}{z} + \sum_{n=0}^{+\infty} b_n z^n,$$

let $(f * g)(z)$ be the Hadamard product or convolution of $f(z)$ and $g(z)$ defined by

$$(f * g)(z) = \frac{1}{z} + \sum_{n=0}^{+\infty} a_n b_n z^n. \tag{1.3}$$

The generalized hypergeometric function ${}_lF_m$ for $a_1,\ \cdots,\ a_l,\ d_1,\ \cdots,\ d_m$ such that $d_j \neq 0, -1, \cdots$ for $j = 1, 2, \cdots, m$, and $z \in \mathbf{C}$ is defined in [2] as follows:

$${}_lF_m(a_1,\ \cdots,\ a_l;\ d_1,\ \cdots,\ d_m;\ z) = \sum_{n=0}^{+\infty} \frac{(a_1)_n \cdots (a_l)_n z^n}{(d_1)_n \cdots (d_m)_n n!} \tag{1.4}$$

with $l \leq m+1$, $l, m \in \mathbf{N}$, where the Pochhammer symbol $(\nu)_n$ (or the shifted factorial since $(1)_n = n!$) is given in terms of the gamma function as

$$(\nu)_n = \frac{\Gamma(\nu+n)}{\Gamma(\nu)} = \begin{cases} 1, & n = 0,\ \nu \in \mathbf{C} - \{0\}); \\ \nu(\nu+1)\cdots(\nu+n-1), & n \in \mathbf{N}_+,\ \nu \in \mathbf{C}). \end{cases}$$

For the positive real values $a_1,\ \cdots,\ a_l,\ d_1,\ \cdots,\ d_m$ such that $d_j \neq 0, -1, \cdots$ for $j = 1, 2, \cdots, m$, by using the Gaussian hypergeometric function given by (1.4), we thus obtain

$${}_lI_m f(z) = z^{-1}({}_lF_m(a_1,\ \cdots,\ a_l;\ d_1,\ \cdots,\ d_m;\ z)) * f(z) = \frac{1}{z} + \sum_{n=0}^{+\infty} \phi_n a_n z^n, \tag{1.5}$$

where

$$\phi_n = \frac{\prod\limits_{i=1}^{l} (a_i)_{n+1}}{\prod\limits_{i=1}^{m} (d_i)_{n+1} (n+1)!} \tag{1.6}$$

(see [3]–[5], and also the more recent works [6]–[8] dealing extensively with Dziok-Srivastava operator).

We note that:

(i) The differential operator ${}_2I_1(a,\ b;\ c;\ z) = (I_c^{a,b} f)(z)$ $(a, b \in \mathbf{C},\ c \in \mathbf{Z}^+)$ was studied by Hohlov[9];

(ii) The differential operator ${}_2I_1(n+1,\ 1;\ 1;\ z) = D^n f(z)$ $(n \in \mathbf{N}^+)$ was studied by Ruscheweyh[10];