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A Kind of Boundary Value Problems for Stochastic Differential Equations

Hu Ling, Wu Zheng, Wei Zhang-zhi and Wang Liang-long*
(*School of Mathematical Sciences, Anhui University, Hefei, 230601*)

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Abstract: In this paper we discuss stochastic differential equations with a kind of periodic boundary value conditions (in sense of mean value). Appealing to the decomposition of equations, the existence of solutions is obtained by using the contraction mapping principle and Leray-Schauder fixed point theorem, respectively.
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1 Introduction

As we all know, boundary value problems (BVPs) for differential equations are very important in applications. In these years, a lot of existence of solutions for BVPs of deterministic differential equations have been obtained (see [1]–[4]), but there is much less for stochastic differential equations.

In 1991, Nualart and Pardoux[5]–[6] discussed the first and second order stochastic differential equations under corresponding conditions:

$$\begin{cases} \dfrac{\mathrm{d}X_t}{\mathrm{d}t} + f(X_t) = B\dfrac{\mathrm{d}W_t}{\mathrm{d}t}, \\ h(X_0, X_1) = \overline{h} \end{cases}$$

and

$$\begin{cases} \dfrac{\mathrm{d}^2 X_t}{\mathrm{d}t^2} + f\left(X_t, \dfrac{\mathrm{d}X_t}{\mathrm{d}t}\right) = \dfrac{\mathrm{d}W_t}{\mathrm{d}t}, \\ X_0 = a, \qquad X_1 = b. \end{cases}$$

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* **Corresponding author**.
E-mail address: huling@hsu.edu.cn (Hu L), wangll@ahu.edu.cn (Wang L L).

Their method was similar to the method of variation of constants.

In 2009, by using the Green's function and the fixed point theorem, Dhage and Badgire[7] got the existence of solution of the following equations:

$$\begin{cases} -x''(t,\,\omega) = f(t,\,x(t,\,\omega),\,\omega), & \text{a.e. } t \in J, \\ x(0,\,\omega) = x(2\pi,\,\omega), & x'(0,\,\omega) = x'(2\pi,\,\omega). \end{cases}$$

They discussed the random equation for existence as well as for existence of extremal solutions under suitable conditions of the nonlinearity f which thereby generalizes several existence results.

Fedchenko and Prigarin[8] investigated stationary boundary value problem

$$\begin{cases} \mathrm{d}x(t) = A(t)x(t)\mathrm{d}t + \sum(t)\mathrm{d}w(t), \\ x(\alpha) = x_\alpha, \qquad x(\beta) = x_\beta. \end{cases}$$

The relative simplicity of linear problems studied allows one to obtain complete results on the existence and uniqueness of the solution, formulate the equivalent Cauchy problem, find a link with deterministic optimal control problems, and solve a number of other problems.

Cao *et al.*[9] in 2014 paid their attention to the numerical solutions as follows:

$$\begin{cases} \dfrac{\mathrm{d}^2u}{\mathrm{d}t^2} + f\left(u,\,\dfrac{\mathrm{d}u}{\mathrm{d}t}\right) = \dfrac{\mathrm{d}W(t)}{\mathrm{d}t} \\ u(0) = a, \qquad u(1) = b. \end{cases}$$

It was proved that under certain regularity conditions, the resulting numerical solution of the homotopy method converged at the same rate as the numerical algorithm used to solve the initial value problem. Their numerical experiments demonstrate that the homotopy continuation method may be less restrictive in selecting the initial iterative point than the shooting method. It should be noted that the convergence analysis of the shooting method was incomplete and the method developed in this paper can be used to obtain the convergence rate of the shooting method.

The above BVPs are deterministic, but in many models the boundary value conditions are indeterminate. Actually there is not much research in this area. Fortunately Wang and Han[10] discussed the periodic boundary value problem as follows:

$$\begin{cases} \mathrm{d}x = f(t,\,x)\mathrm{d}t + g(t,\,x)\mathrm{d}W, \\ Ex(0) = Ex(T). \end{cases} \tag{1.1}$$

Under the suitable conditions, the existence of solution was obtained by a new technique. The main idea is to decompose the stochastic process into a deterministic term and a new stochastic term with zero mean value.

It is easily known that the BVPs of stochastic differential equation (SDE) are of great sense in many fields. In this paper, we discuss the problem as follows:

$$\begin{cases} \mathrm{d}x = f(x,\,t)\mathrm{d}t + g(x,\,t)\mathrm{d}W, \\ Ex(0) + Ex(T) = \alpha Ex\left(\dfrac{T}{2}\right), \end{cases} \tag{1.2}$$

where α is a constant, especially when $\alpha = 2$, $Ex\left(\dfrac{T}{2}\right)$ is the mean value of $Ex(0)$ and $Ex(T)$. In order to get the solution $x(t)$, we decompose it with

$$x(t) = y(t) + z(t).$$