COMMUNICATIONS
IN
MATHEMATICAL
RESEARCH
34(3)(2018), 221–229

Discrete Arzelà-Ascoli Theorem on the Half Line and Its Application

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Communicated by Shi Shao-yun

Abstract: The compactness criterion in $l_\diamond = \left\{x(\,\cdot\,) \in l^\infty(0,\infty)) \mid \lim\limits_{t\to\infty} x(t) = x(\infty)\right\}$ is given and as an application, we study the existence of positive solutions of second order boundary value problems of difference equation on the half line by the fixed-point theorem in cones.

Key words: discrete Arzelà-Ascoli theorem, half line, positive solution, difference equation, fixed point

2010 MR subject classification: 34G20

Document code: A

Article ID: 1674-5647(2018)03-0221-09

DOI: 10.13447/j.1674-5647.2018.03.04

1 Introduction

The well-known Arzelà-Ascoli theorem gives necessary and sufficient conditions to decide whether every subsequence of a given sequence of real-valued continuous functions defined on a closed and bounded interval has a uniformly convergent subsequence. The theorem is a fundamental result in functional analysis. In particular, it forms the basis for the proof of the Peano existence theorem in the theory of ordinary differential equations (see [1]) and Montel's theorem in complex analysis (see [2]). It also plays a decisive role in the proof of the Peter-Weyl theorem (see [3]).

First we briefly recall some history about Arzelà-Ascoli theorem. The notion of equicontinuity was introduced at around the same time by Arzelà[4] and Ascoli[5]. A weak form of the theorem was proven by Ascoli[5], who established the sufficient condition for compactness, and by Arzelà[6], who established the necessary condition and gave the first clear presentation of the result. A further generalization of the theorem was proven by Fréchet[7] to sets of real-valued continuous functions with domain a compact metric space, see Dunford

Received date: Feb. 16, 2017.
Foundation item: The NSF (11626188, 11671322) of China, Gansu Provincial NSF (1606RJYA232) of China and NWNU-LKQN-15-16.
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and Schwartz's work (see [8]). Modern formulations of the theorem allow for the domain to be compact Hausdorff and for the range to be an arbitrary metric space. More general formulations of the theorem exist that give necessary and sufficient conditions for a family of functions from a compactly generated Hausdorff space into a uniform space to be compact in the compact-open topology[9]. Later, many mathematicians, such as Corduneanu[10], Kelley and Namioka[11], Kelley[9] developed their work. The investigation of compactness criterion has involved much interest in the new century (see [10]–[12]). The compact criterion in

$$C_l = \left\{ f \in C([0,\infty)) \mid \lim_{t\to\infty} f(t) = r \text{ for some } r \in \mathbf{R} \right\}$$

has been investigated in Chapter 2 of [10], the result is as follows:

Theorem A *Let $\mathcal{F} \subset C_l$ be a set satisfying the following conditions*:

(i) *$\mathcal{F}$ is bounded in C_l*;

(ii) *the function belonging to $\mathcal{F}$ are equicontinuous on any compact interval of* $[0, \infty)$.

(iii) *the function from $\mathcal{F}$ are equiconvergent, i.e., given $\varepsilon > 0$, there corresponds $T(\varepsilon) > 0$ such that $\|f(t) - f(\infty)\| < \varepsilon$ for any $t \geq T(\varepsilon)$ and $f \in \mathcal{F}$.*
Then $\mathcal{F}$ is compact in C_l.

It has been pointed out that Theorem A plays an important role to study the positive solutions of second order boundary value problem of ordinary differential equation on the half line by the fixed-point theorem in cones (see [13]–[15] and their references). However, there is few literature to study the boundary value problems of difference equation on the half line by the fixed-point theorem in cones because of no compact criterion in

$$l_\diamond = \left\{ x(\,\cdot\,) \in l^\infty(0,\ \infty) \,\middle|\, \lim_{n\to\infty} x(n) = x(\infty) \right\}.$$

A natural question is whether exists the compactness criterion in $l_\diamond$? In this paper, we give a positive answer and obtain the necessary and sufficient conditions of the compactness criterion in $l_\diamond$. It is particularly useful to study the boundary value problems of difference equation on the half line.

The rest of this paper is organized as follows. In Section 2, we present the discrete Arzelà-Ascoli Theorem on the half line. Section 3 is devoted to the application of the discrete Arzelà-Ascoli Theorem.

2 Compactness Results

Let $\mathbf{N}^+ = \{1, 2, \cdots\}$, $\mathbf{N} = \{0, 1, 2, \cdots\}$ and $[a, b]_\mathbf{Z} = \{a, a+1, a+2, \cdots, b\}$ with $a < b$, $a, b \in \mathbf{N}^+$.

Let

$$l_\diamond = \left\{ x(\,\cdot\,) \in l^\infty(0,\ \infty) \,\middle|\, \lim_{t\to\infty} x(t) = x(\infty) \right\}$$

with the norm

$$\|x\|_\infty = \sup_{t\in\mathbf{N}} |x(t)|.$$

Then $l_\diamond$ is a Banach space.