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The Twisted Transfer Variety

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Abstract: In this paper, we describe the variety defined by the twisted transfer ideal. It turns out that this variety is nothing but the union of reflecting hyperplanes and the fixed subspaces of the elements of order p in G.

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1 Introduction

Let V be a vector space of dimension n over a field k, G a finite linear group of V. The induced action on the dual V^* extends to the polynomial ring $k[V]$ which is given by

$$(gf)(v) = f(g^{-1}v), \qquad g \in G,\ f \in k[V],\ v \in V.$$

The ring of invariants of G is the subring of $k[V]$ given by

$$k[V]^G := \{f \in k[V] \mid g \cdot f = f,\ g \in G\}.$$

The transfer defined by

$$\mathrm{Tr}^G\colon k[V] \to k[V]^G, \qquad f \mapsto \sum_{g\in G} g \cdot f$$

is a $k[V]^G$-module homomorphism. It is surjective if and only if the characteristic of k does not divide the order of G, i.e., in the nonmodular case. It provides a tool for constructing the ring of invariants. When the characteristic of k is a prime p and divides the order of G, i.e., in the modular case, the image of the transfer, $\mathrm{Im}(\mathrm{Tr}^G)$, is a proper non-zero ideal in $k[V]^G$ (see Theorem 2.2 of [1]). We may extend $\mathrm{Im}(\mathrm{Tr}^G)$ to an ideal $(\mathrm{Im}(\mathrm{Tr}^G))^e$ in $k[V]$ in the usual way. This extended ideal is also called the trace ideal and defines an algebraic set.

Definition 1.1[2] *Let $\rho\colon G \hookrightarrow \mathrm{GL}(n, k)$ be a representation of a finite group over the field*

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k. The transfer variety, denoted by $\mathcal{V}((\mathrm{Im}(\mathrm{Tr}^G))^e) \subseteq V$, is defined by

$$\mathcal{V}((\mathrm{Im}(\mathrm{Tr}^G))^e) = \{x \in V \mid \mathrm{Tr}^G(f)(x) = 0,\ f \in \mathrm{Tot}(k[V])\},$$

where $\mathrm{Tot}(k[V]) = \oplus k[V]_i$ *denotes the totalization of* $k[V]$ *throwing away the grading.*

Remark 1.1 Since $\mathcal{V}((\mathrm{Im}(\mathrm{Tr}^G))^e)$ is an affine variety, we must use all polynomial functions to define it, and not just homogeneous ones.

The trace ideal defines exactly the wild ramification locus as shown first by Auslander and Rim[3] and first noted in the context of invariant theory by Feshbach[4]. Kuhnigk and Smith[5] reproved Feshbach's result and in addition described the transfer variety in the unpublished manuscript. It turns out that this variety has a particularly elegant description. Namely, it is the union of the fixed point sets of the elements of order p in G, where p is the characteristic of k. We refer the reader to [1] and [2] for the transfer variety.

Broer[6] defined a twisted transfer map and gave a characterisation of the direct summand property in terms of the image of twisted transfer maps. The direct summand property holds if and only if the image of the twisted transfer is a cyclic graded $k[V]^G$-module. He conjectured that the invariant ring $k[V]^G$ is a polynomial ring if G is generated by pseudoreflections, and $k[V]^G$ is a direct summand of $k[V]$. It seems worthwhile therefore to study the twisted transfer in modular invariant theory. This is what we do in this article.

We begin this paper by introducing the twisted transfer ideal. We then go on to describe the twisted transfer variety and obtain a decomposition of the radical of the twisted transfer ideal.

2 The Twisted Transfer Ideal

Let $R \subseteq S$ be an integral extension of integral domains such that the extension of their quotient fields $K \subseteq L$ is finite and separable. Multiplication by an element $y \in L$ is a linear map of the finite dimensional K-vector space L. We write its trace as $\mathrm{Tr}_{L/K}(y)$. Then the symmetric K-bilinear form

$$L \times L \to K, \qquad (y_1, y_2) \mapsto \mathrm{Tr}_{L/K}(y_1 y_2)$$

is nondegenerate (see Lemma 3.7.2 of [7]). Classically, the (Dedekind) inverse different or complementary module is defined by

$$(\mathfrak{D}^D_{S/R})^{-1} := \{y \in L \mid \mathrm{Tr}_{L/K}(Sy) \subseteq R\}.$$

The (Dedekind) different is then

$$\mathfrak{D}^D_{S/R} := \{y \in L \mid y(\mathfrak{D}^D_{S/R})^{-1} \subseteq S\}.$$

Broer[8] defined the twisted trace ideal as

$$\mathfrak{D}^T_{S/R} := \mathfrak{D}^D_{S/R} \cdot \mathrm{Tr}((\mathfrak{D}^D_{S/R})^{-1}).$$

In invariant theory, stronger hypotheses are satisfied for our extensions, so we are able to say more. In particular, we restrict ourselves to the following situation. Let $K \subseteq L$ be a Galois extension with Galois group G. Then