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Hermite-Hadamard Type Fractional Integral Inequalities for Preinvex Functions

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Abstract: In this article, we extend some estimates of the right-hand side of the Hermite-Hadamard type inequality for preinvex functions with fractional integral. The notion of logarithmically s-Godunova-Levin-preinvex function in second sense is introduced and then a new Hermite-Hadamard inequality is derived for the class of logarithmically s-Godunova-Levin-preinvex function.

Key words: Hermite-Hadamard's integral inequality, Riemann-Liouville fractional integral, Hölder's integral inequality

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1 Introduction

Let $\mathbf{R}$ be the set of real numbers, $I \subset \mathbf{R}$, I^o is the interior of I. It is common knowledge in mathematical analysis that a function $f\colon I \subset \mathbf{R} \to \mathbf{R}$ is said to be convex on an interval I if the inequality

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y) \tag{1.1}$$

is valid for all $x, y \in I$ and $\lambda \in [0, 1]$.

Many inequalities have been established for convex functions but the most famous is the Hermite-Hadamard's integral inequality, due to its rich geometrical significance and applications, which is stated as follow (see [1]):

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If $f\colon I \subset \mathbf{R} \to \mathbf{R}$ is a convex function on I and $a, b \in I$ with $a < b$, then the double inequalities

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \le \frac{f(a)+f(b)}{2} \tag{1.2}$$

hold.

In [2], Dragomir and Agarwal proved the following results connected with the right part of (1.2).

Lemma 1.1[2] *Let $f\colon I^o \subseteq \mathbf{R} \to \mathbf{R}$ be a differentiable mapping on I^o, $a, b \in I^o$ with $a < b$. If $f' \in L[a, b]$, then the following equality holds*:

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x = \frac{b-a}{2}\int_0^1 (1-2t)f'(ta+(1-t)b)\mathrm{d}t. \tag{1.3}$$

Theorem 1.1[2] *Let $f\colon I^o \subseteq \mathbf{R} \to \mathbf{R}$ be a differentiable mapping on I^o, $a, b \in I^o$ with $a < b$. If $|f'|$ is convex on $[a, b]$, then the following inequality holds*:

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \le \frac{b-a}{8}(|f'(a)|+|f'(b)|) \tag{1.4}$$

In [3], Sarikaya *et al.* established Hermite-Hadamard's inequalities for Riemann-Liouville fractional integral. And furthermore, (1.3) and (1.4) for fractional integral type were all obtained.

Fractional calculus is a theory of integral and differential operators of non-integral order. Many mathematicians, like Liouville, Riemann and Weyl, made major contributions to the theory of fractional calculus. The study on the fractional calculus continued with the contributions from Fourier, Abel, Lacroix, Leibniz, Grunwald and Letnikov. For details, see [4]–[6]. A first formulation of an integral operator of fractional order in reliable form is named the Riemann-Liouville fractional integral operator.

In recent years, several extensions and generalizations have been considered for classical convexity. A significant generalization of convex functions is that of invex functions introduced by Hanson[7]. Weir and Mond[8] introduced the concept of preinvex functions and applied it to the establishment of the sufficient optimality conditions and duality in nonlinear programming. Noor[9],[10] introduced the Hermite-Hadamard inequality for preinvex and log-preinvex functions.

In this paper, we promote all the results of literature [3] for preinvex functions. Then we need the following definitions. For more details, one can consult [4]–[17].

Definition 1.1[3] *Let $f \in L_1[a, b]$. The Riemann-Liouville integrals $J_{a^+}^{\alpha} f$ and $J_{b^-}^{\alpha} f$ of order $\alpha > 0$ with $a \ge 0$ are defined by*

$$J_{a^+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)\mathrm{d}t, \qquad x > a,$$

$$J_{b^-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f(t)\mathrm{d}t, \qquad x < b,$$