COMMUNICATIONS IN MATHEMATICAL RESEARCH
34(4)(2018), 371–382

Boundedness of Marcinkiewicz Integrals in Weighted Variable Exponent Herz-Morrey Spaces

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Communicated by Ji You-qing

Abstract: In this paper, under natural regularity assumptions on the exponent function, we prove some boundedness results for the functions of Littlewood-Paley, Lusin and Marcinkiewicz on a new class of generalized Herz-Morrey spaces with weight and variable exponent, which essentially extend some known results.

Key words: Marcinkiewicz integral, variable exponent, Muckenhoupt weight, Herz-Morrey space

2010 MR subject classification: 46E30, 42B35

Document code: A

Article ID: 1674-5647(2018)04-0371-12

DOI: 10.13447/j.1674-5647.2018.04.10

1 Introduction

Suppose that $\mathbb{S}^{n-1}$ is the unit sphere in $\mathbf{R}^n$ $(n \geq 2)$ equipped with the normalized Lebesgue measure $\mathrm{d}\sigma(x')$. Let $\Omega \in L^1(\mathbb{S}^{n-1})$ be homogeneous of degree zero and satisfy

$$\int_{\mathbb{S}^{n-1}} \Omega(x')\mathrm{d}\sigma(x') = 0, \tag{1.1}$$

where $x' = \dfrac{x}{|x|}$ for any $x \neq 0$. Then the n-dimension Marcinkiewicz integral operator μ is defined by

$$\mu(f)(x) = \left(\int_0^\infty |F_t(f)(x)|^2 \frac{\mathrm{d}t}{t^3}\right)^{\frac{1}{2}},$$

where

$$F_t(f)(x) = \int_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y)\mathrm{d}y.$$

Received date: March 4, 2018.

Foundation item: The NSF (11771223) of China.

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It is well known that by means of the real variables method, Stein[1] first proved that if Ω satisfies a Lip_γ $(0 < \gamma \le 1)$ condition on $\mathbb{S}^{n-1}$, that is,

$$|\Omega(x') - \Omega(y')| \le C|x' - y'|^\gamma, \qquad x',\, y' \in \mathbb{S}^{n-1}, \tag{1.2}$$

then μ is of type (p, p) for $1 < p \le 2$ and of weak type $(1, 1)$. Using a good-λ inequality, Torchinsky and Wang[2] improved the boundedness of μ to the spaces $L^p(\omega)$ provided that $\omega \in A_p$ for $1 < p < \infty$ (see Section 2.2 for the definition of the A_p condition). Subsequently, Al-Salman, Ding, *et al.* made important progress on this operator, we refer to [3]–[5] for some recent development.

In recent years, following the fundamental work of Kováčik and Rákosník[6], function spaces with variable exponent have attracted a great attention in connection with problems of the boundedness of classical operators on those spaces, which in turn were motivated by the treatment of recent problems arising in PDEs and the calculus of variations, see [7]–[17] and the references therein.

The classical theory of Muckenhoupt A_p weights is a powerful tool in harmonic analysis, for example in the study of boundary value problems for Laplace's equation on Lipschitz domains. Recently, the generalized Muckenhoupt $A_{p(\cdot)}$ weights with variable exponent $p(\cdot)$ have been intensively studied by many authors (see [18] and [19]). In particular, we note that Diening and Hästö[20] introduced the new class of $A_{p(\cdot)}$ weights and proved the equivalence between the $A_{p(\cdot)}$ condition and the boundedness of the Hardy-Littlewood maximal operator M on weighted $L^{p(\cdot)}$ spaces. Cruz-Uribe and Wang[19] extended the theory of Rubio de Francia extrapolation to the weighted variable Lebesgue spaces $L^{p(\cdot)}(\omega)$. As a consequence they showed that a number of different operators, such as Calderón-Zygmund singular integral operator, fractional integral operator and elliptic operator in divergence form *etc.*, are bounded on these spaces.

Herz spaces $\dot{K}^{\alpha,q}_{p(\cdot)}(\omega)$ and $K^{\alpha,q}_{p(\cdot)}(\omega)$ with variable exponent $p(\cdot)$ and $A_{p(\cdot)}$ weights, namely, weighted variable exponent Herz spaces, were introduced first by Izuki and Noi[21],[22]. Using the basics on Banach function spaces and the Muckenhoupt theory with variable exponent, they established the boundedness of an intrinsic square function and fractional integral operators on such spaces. The class of variable exponent Herz-Morrey spaces $M\dot{K}^{\alpha,\lambda}_{q,p(\cdot)}(\mathbf{R}^n)$ is defined by Izuki[23], and the boundedness of sublinear operators satisfying a proper size condition on $M\dot{K}^{\alpha,\lambda}_{q,p(\cdot)}(\mathbf{R}^n)$ is obtained. Furthermore, Izuki[24] studied the well known Hardy-Littlewood-Sobolev theorem on $M\dot{K}^{\alpha,\lambda}_{q,p(\cdot)}(\mathbf{R}^n)$, which generalized the one obtained by Lu and Yang[25] in the classical Herz spaces.

Motivated by the results mentioned above, the principal problem considered in this paper is to define weighted variable exponent Herz-Morrey spaces and study the boundedness of the functions of Littlewood-Paley, Lusin and Marcinkiewicz on these spaces. We note that our main results (see Theorems 3.1 and 3.2 below) improve and extend the corresponding main theorems in Torchinsky and Wang[2], where the constant exponent case was studied.

In general, by B we denote the ball with center $x \in \mathbf{R}^n$ and radius $r > 0$. If E is a subset of $\mathbf{R}^n$, $|E|$ denotes its Lebesgue measure and χ_E its characteristic function. $p'(\cdot)$ denotes