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Fundamental Solution of Dirichlet Boundary Value Problem of Axisymmetric Helmholtz Equation

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Abstract: Fundamental solution of Dirichlet boundary value problem of axisymmetric Helmholtz equation is constructed via modified Bessel function of the second kind, which unified the formulas of fundamental solution of Helmholtz equation, elliptic type Euler-Poisson-Darboux equation and Laplace equation in any dimensional space.
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1 Introduction

In this paper, we study fundamental solution of Dirichlet boundary value problem of axisymmetric Helmholtz equation in the upper half space

$$\begin{cases} \partial_t^2 u + \Delta_x u + \dfrac{\alpha}{t}\partial_t u + \lambda^2 u = 0 & \text{in } \mathbf{R}_+^{n+1}, \\ u(0, x) = \delta(x) & \text{in } \mathbf{R}^n, \\ u(+\infty, x) \text{ is bounded}, \end{cases} \tag{1.1}$$

where

$$\mathbf{R}_+^{n+1} = \{(t, x) : t > 0, x \in \mathbf{R}^n\},$$

and real-valued parameters $\alpha < 1$ and $\lambda > 0$.

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This problem is closely connected with the study of electromagnetic scattering (see [1]–[2]). During considering the processes taking place in some inhomogeneous media with fractal structure one also must take into account fluctuations of the parameter's value. For example, the parameter α in (1.1) has the sense of Hausdorff dimension when studying the probability density function in the capacity of unknown (see [3]).

For the case $\alpha = 0$ and $\lambda = 0$, (1.1) is the classic Laplace equation. When $\alpha = 0$ and $\lambda > 0$, (1.1) becomes the classic Helmholtz equation, this is the reason that it is called axisymmetric Helmholtz equation.

The method of fundamental solutions plays an important role in study of partial differential equations. In [4], the potentials were constructed via fundamental solutions of Helmholtz equation. Analogously to the potential theory for Helmholtz equation, one can construct a potential for axisymmetric Helmholtz equation whose kernels are written via fundamental solutions of axisymmetric Helmholtz equation. Various regularization approaches for the study of the method of fundamental solutions were studied in [5]. In this scope, the method of fundamental solutions generated by classic Helmholtz operator and axisymmetric Helmholtz equation with $\alpha > 0$ in different dimensional space were also investigated, see [6]–[10] and the references therein. Thus, the case of $\alpha < 0$ and $\lambda > 0$ might become an object of a new research.

When $\lambda = 0$, (1.1) is elliptic type generalized Euler-Poisson-Darboux equation whose fundamental solutions was established by similarity method in [11]. Motivated by this, we construct fundamental solution

$$P(t,\, x,\, \alpha,\, \lambda) = \frac{(\mathrm{i}\lambda)^{\frac{1+n-\alpha}{2}}}{2^{\frac{n-\alpha-1}{2}}\pi^{\frac{n}{2}}\Gamma\Big(\dfrac{1-\alpha}{2}\Big)} \cdot \frac{t^{1-\alpha}K_{\frac{1+n-\alpha}{2}}(\mathrm{i}\lambda\sqrt{t^2+|x|^2})}{(\sqrt{t^2+|x|^2})^{\frac{1+n-\alpha}{2}}} \tag{1.2}$$

and then solves (1.1) in general sense in the upper half space. In particular, the explicit formula of $P(t,\, x,\, \alpha,\, \lambda)$ is not restricted by dimensional numbers.

2 Construction of Fundamental Solution

In [11], the fundamental solution $P(x,\, y,\, \alpha,\, 0)$ of (1.1) was constructed by use of similarity method

$$P(t, x, \alpha, 0) = \frac{\Gamma\Big(\dfrac{1+n-\alpha}{2}\Big)}{\pi^{\frac{n}{2}}\Gamma\Big(\dfrac{1-\alpha}{2}\Big)}\frac{t^{1-\alpha}}{(t^2+|x|^2)^{\frac{1+n-\alpha}{2}}}, \tag{2.1}$$

where $\Gamma(z)$ is Gamma function. In this section, we seek a fundamental solution in the same form

$$P(x,\, y,\, \alpha,\, \lambda) = C(\alpha,\, \lambda)t^{1-\alpha}p(t^2+|x|^2)$$

for the case $\lambda > 0$, where the constant $C(\alpha,\, \lambda)$ will be determined in the following.

Set

$$r = |x|, \qquad s = t^2 + r^2, \qquad u(t, x) = t^{1-\alpha}v(s).$$