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On a Generalized Matrix Algebra over Frobenius Algebra

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Abstract: Let A be a Frobenius k-algebra. The matrix algebra $R = \begin{pmatrix} A & {}_AA_k \\ {}_kA_A & k \end{pmatrix}$ is called a generalized matrix algebra over a Frobenius algebra A. In this paper we show that R is also a Frobenius algebra.

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1 Introduction

Let A, B be two finite dimensional algebras over a field k, ${}_AM_B$, ${}_BN_A$ be two finitely generated bimodules. Assume that there are bimodule morphisms

$$\tau\colon M \otimes_B N \longrightarrow A\colon \tau(m \otimes n) = (m, n)$$
$$\mu\colon N \otimes_A M \longrightarrow B\colon \mu(n \otimes m) = [n, m]$$

satisfying

$$(m, n)m' = m[n, m'], \quad [n, m]n' = n(m, n'), \qquad m, m' \in M,\ n, n' \in N,$$

where addition and multiplication are defined as in customary for matrices, $R = \begin{pmatrix} A & M \\ N & B \end{pmatrix}$ is a k-algebra, which is called generalized matrix algebra. One point extension algebra and local extension algebra are also generalized matrix algebras. Generalized matrix algebra also comes up as a Morita Context. For more details see [1]–[3].

Frobenius bimodules are connected with Frobenius algebras and extensions. For instance, a ring extension $\phi\colon R \to S$ is a Frobenius extension if and only if ${}_RS_S$ is a Frobenius

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bimodule. Let A be a finite dimensional k-algebra. If ${}_kA_A$ is a Frobenius bimodule, there exists a bimodule isomorphism $\mathrm{Hom}_k({}_kA,\, k) \cong {}_AA_k$, then A is called a Frobenius algebra. Simple algebra over a field k, group algebra kG are also Frobenius algebra. By [1] (see p261), if A is a Frobenius algebra, then

$$\tau\colon {}_AA \otimes_k A_A \longrightarrow A, \qquad \mu\colon {}_kA \otimes_A k \longrightarrow A_k.$$

So $R = \begin{pmatrix} A & {}_AA_k \\ {}_kA_A & k \end{pmatrix}$ is a generalized matrix algebra over a Frobenius algebra A. In present paper, we show that R is also a Frobenius algebra. Throughout this paper, all rings have an identity element and all modules are unital, the following symbols can be referred in [4]–[6]. The latest related research on this subject can be found in [7]–[11].

2 The Functor Between mod-$A \times B$ and mod-R

For a ring A, the category of left A-modules is denoted by A-mod, mod-A denotes the category of the right A-modules. Let $\mathcal{A}(R)$ be the category whose objects are $(X,\, Y)_{\alpha,\,\beta}$, where $X \in$ mod-A, $Y \in$ mod-B, $\alpha \in \mathrm{Hom}_B(X \otimes_A M,\, Y)$, $\beta \in \mathrm{Hom}_A(Y \otimes_B N,\, X)$ such that

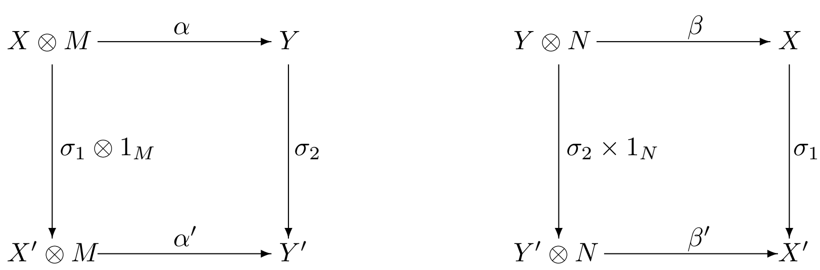
$$\alpha(\beta(y \otimes n) \otimes m) = y[n,\, m], \qquad \beta(\alpha(x \otimes m) \otimes n) = x(m,\, n)$$

for all $x \in X$, $y \in Y$, $m \in M$, $n \in N$.

Instead of α and β, it is more convenient to use the following homomorphisms $\overline{\alpha}$ and $\overline{\beta}$,

$$\overline{\alpha}\colon X \to \mathrm{Hom}_B(M,\, Y), \qquad \overline{\alpha}(x)m = \alpha(x \otimes m),$$
$$\overline{\beta}\colon Y \to \mathrm{Hom}_A(N,\, X), \qquad \overline{\beta}(y)n = \beta(y \otimes n).$$

The morphisms of $\mathcal{A}(R)$ are pairs of $(\sigma_1,\, \sigma_2)$, where $\sigma_1 \in \mathrm{Hom}_A(X,\, X')$, $\sigma_2 \in \mathrm{Hom}_B(Y,\, Y')$ such that the following diagrams are commutative.



Green[4] proved that the category $\mathcal{A}(R)$ is equivalent to the category mod-R, i.e., there exists a categorical equivalent functor

$$F\colon \mathcal{A}(R) \Leftrightarrow \text{mod-}R$$

such that

$$F(X,\, Y)_{\alpha,\beta} = X \oplus Y,$$

where the right modular operation is

$$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} a & m \\ n & b \end{pmatrix} = \begin{pmatrix} xa + \beta(y \otimes n) & \alpha(x \otimes n) + yb \end{pmatrix}.$$