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Singly Covered Minimal Elements of Linked Partitions and Cycles of Permutations

Guo Wen-kui and Niu Fei-fei
(*School of Science, Beijing Technology and Business University, Beijing, 100048*)

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Abstract: Linked partitions were introduced by Dykema (Dykema K J. Multilinear function series and transforms in free probability theory. *Adv. Math.*, 2005, 208(1): 351–407) in the study of the unsymmetrized T-transform in free probability theory. Permutation is one of the most classical combinatorial structures. According to the linear representation of linked partitions, Chen *et al.* (Chen W Y C, Wu S Y J, Yan C H. Linked partitions and linked cycles. *European J. Combin.*, 2008, 29(6): 1408–1426) defined the concept of singly covered minimal elements. Let $L(n, k)$ denote the set of linked partitions of $[n]$ with k singly covered minimal elements and let $P(n, k)$ denote the set of permutations of $[n]$ with k cycles. In this paper, we mainly establish two bijections between $L(n, k)$ and $P(n, k)$. The two bijections from a different perspective show the one-to-one correspondence between the singly covered minimal elements in $L(n, k)$ and the cycles in $P(n, k)$.

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1 Introduction

Linked partitions were introduced by Dykema[1] in the study of the unsymmetrized T-transform in free probability theory. Let $[n] = \{1, 2, \cdots, n\}$. A linked partition (see [2]) of $[n]$, is a collection of nonempty subsets $B_1, B_2, \cdots, B_k$ of $[n]$, called blocks, such that the union of $B_1, B_2, \cdots, B_k$ is $[n]$ and any two distinct blocks are nearly disjoint. Two blocks B_i and B_j are said to be nearly disjoint if for any $k \in B_i \cap B_j$, one of the following conditions holds:

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E-mail address: guowkmath@163.com (Guo W K).

(1) $k = \min\{B_i\}$, $|B_i| > 1$ and $k \neq \min\{B_j\}$, or

(2) $k = \min\{B_j\}$, $|B_j| > 1$ and $k \neq \min\{B_i\}$.

The linear representation of a linked partition was introduced by Chen *et al.*[2]. For a linked partition τ of $[n]$, list n vertices in a horizonal line with labels $1, 2, \cdots, n$. For each block $B = \{i_1, i_2, \cdots, i_k\}$ with $k \geq 2$ and $i_1 < i_2 < \cdots < i_k$, we draw an arc from i_1 to i_j for any $j = 2, \cdots, k$. For $i < j$, we use a pair (i, j) to denote an arc from i_1 to i_j, and we call i and j the left-hand endpoint and the right-hand endpoint of (i, j), respectively. Chen *et al.*[3] introduced a classification of vertices in the linear representation of a linked partition as follows: For any vertex i, $1 \leqslant i \leqslant n$,

1. If i is only a left-hand endpoint, then we call it an origin;
2. If i is both a left-hand point and a right-hand point, then we call it a transient;
3. If i is isolated, then it is called a singleton;
4. If i is only a right-hand endpoint, then it is called a destination.

Fig. 1.1 illustrates a linked partition of [10], where the element 1 is an origin, 2, 3, 5, 6 and 8 are transients, 7 is a singleton and 4, 9 and 10 are destinations.

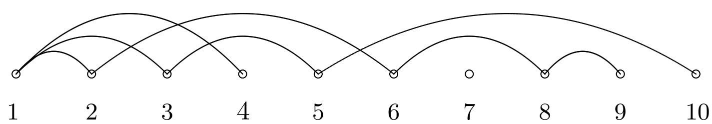

Fig. 1.1 The linear representation of $\tau = \{1, 2, 3, 4\}\{2, 6\}\{3, 5\}\{5, 10\}\{6, 8\}\{7\}\{8, 9\}$

The origins, singletons, and destinations are called singly covered elements, where origins and singletons are also called singly covered minimal elements. The transients are called doubly covered elements. The enumeration of permutations and linked partitions are both $n!$. Corteel[4] gave a graphical representation of permutations, in which there are interesting structures of crossings and nestings. Thanatipanonda[5] described the conceptions of inversions and major index of permutations. Stanley[6] introduced a variety of statistics of permutations, including cycle structures, descents, etc. He also introduced the relationship of cycle structures and Stirling number of the first kind. Define $c(n, k)$ to be the number of permutations with exactly k cycles, and $c(n, k)$ is called a signless Stirling number of the first kind.

In the following sections, we discuss the structural properties of linked partitions and permutations. Specifically, we present two different bijections between linked partitions and permutations such that the one-to-one correspondence between the singly covered minimal elements of linked partitions and the cycles of permutations is evident.

2 Bijection I

In this section, we exhibit our first bijection between linked partitions of n with k singly covered minimal elements and permutations of n with k cycles, which implies some interesting properties between connectivity of linked partitions and the cycles of permutations. Denote the set of linked partitions of n by $L(n)$ and denote the set of linked partitions of