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Hermite-Hadamard Type Inequalities for Operator h-preinvex Functions

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Abstract: Operator h-preinvex functions are introduced and a refinement of Hermite-Hadamard type inequalities for such functions is established. Results proved in this paper are more general and some known results are special cases.

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1 Introduction

Let I, J be intervals in $\mathbf{R}$, $(0, 1) \subseteq J$. Let $B(\mathcal{H})$ be the algebra of all bounded linear operators on a complex separable Hilbert space $\mathcal{H}$. Denoted by $B(\mathcal{H})_{ad}$ the set of selfadjoint operator in $B(\mathcal{H})$. In 1991, Pečcarić *et al.*[1] proved the following integral inequality:

Let $f\colon I \subseteq \mathbf{R} \to \mathbf{R}$ be a convex function and $a, b \in I$ with $a < b$. Then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \leq \frac{f(a)+f(b)}{2}, \tag{1.1}$$

which is known as the Hermite-Hadamard's integral inequality.

More recently, a number of papers have been written providing noteworthy extensions, generalizations and refinements for more extensive functions (see [2]–[15]).

Sarikaya *et al.*[15] introduced a new class of convex functions called h-convex functions, and proved the following Hermite-Hadamard type inequalities for $h-$convex functions.

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Definition 1.1[15] *Let $h\colon J \to \mathbf{R}$ be a non-negative function. We say that $f\colon I \subseteq \mathbf{R} \to \mathbf{R}$ is an h-convex function, if f is nonnegative and for all $x, y \in I$ and $t \in (0, 1)$, we have*

$$f(tx + (1-t)y) \le h(t)f(x) + h(1-t)f(y). \tag{1.2}$$

If the above inequality is reversed, then f is said to be h-concave.

This notion unifies and generalizes the known classes of functions, for instance, convex functions, s-convex functions in the second sense, Gudunova-Levin functions and P-functions, which are obtained by putting in (1.2),

$$h(t) = t, \quad h(t) = t^s, \quad h(t) = \frac{1}{t}, \quad h(t) = 1,$$

respectively. Many properties of functions mentioned above can be found in [12]–[14].

Theorem 1.1[15] *Let $h\colon J \to \mathbf{R}$ be a non-negative function with $h\left(\frac{1}{2}\right) \neq 0$. If f is h-convex with $f \in L_1[a, b]$, then*

$$\frac{1}{2h\left(\frac{1}{2}\right)} f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \le [f(a) + f(b)]\int_0^1 h(t)\mathrm{d}t. \tag{1.3}$$

Hason[16] gave the notion of invexity as significant generalization of classical convexity. Let $\mathcal{X}$ be a real vector space. A set $\mathcal{S} \subseteq \mathcal{X}$ is said to be invex with respect to the map $\eta\colon \mathcal{S} \times \mathcal{S} \to \mathcal{X}$, if for every $x, y \in \mathcal{S}$ and $t \in [0, 1]$,

$$x + t\eta(y, x) \in \mathcal{S}.$$

It is obvious that every convex set is invex with respect to the map $\eta(y, x) = y - x$, but there exist invex sets which are not convex (see [17]).

For every $x, y \in \mathcal{S}$ the η-path P_{xv} joining the points x and $v := x + \eta(y, x)$ is defined as

$$P_{xv} := \{z \mid z = x + t\eta(y, x)\colon t \in [0, 1]\}.$$

The mapping η is said to be satisfies the condition (C) if for every $x, y \in \mathcal{S}$ and $t \in [0, 1]$,

$$\eta(x, y + t\eta(x, y)) = (1-t)\eta(x, y), \qquad \eta(y, y + t\eta(x, y)) = -t\eta(x, y).$$

Note that for every $x, y \in \mathcal{S}$ and $t \in [0, 1]$, if η satisfying condition (C) we have

$$\eta(y + t_2\eta(x, y),\ y + t_1\eta(x, y)) = (t_2 - t_1)\eta(x, y). \tag{1.4}$$

In fact,

$$\begin{aligned}
&\eta(y + t_2\eta(x, y),\ y + t_1\eta(x, y)) \\
={}& \eta(y + t_2\eta(x, y),\ y + t_2\eta(x, y) + (t_1 - t_2)\eta(x, y)) \\
={}& \eta\left(y + t_2\eta(x, y),\ y + t_2\eta(x, y) + \frac{t_1 - t_2}{1 - t_2}\eta(x,\ y + t_2\eta(x, y))\right) \\
={}& \frac{t_1 - t_2}{1 - t_2}\eta(x,\ y + t_2\eta(x, y)) \\
={}& (t_2 - t_1)\eta(x, y).
\end{aligned}$$