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An Optimal Sixth-order Finite Difference Scheme for the Helmholtz Equation in One-dimension

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Abstract: In this paper, we present an optimal 3-point finite difference scheme for solving the 1D Helmholtz equation. We provide a convergence analysis to show that the scheme is sixth-order in accuracy. Based on minimizing the numerical dispersion, we propose a refined optimization rule for choosing the scheme's weight parameters. Numerical results are presented to demonstrate the efficiency and accuracy of the optimal finite difference scheme.

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1 Introduction

In this paper, we consider the 1D Helmholtz problem (see [1])

$$\mathcal{L}u := -\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} - k^2 u = f \tag{1.1}$$

with the wavenumber k, where unknown u usually represents a pressure field in the frequency domain, and f denotes the source function. The Helmholtz equation has important applications in acoustic, electromagnetic wave scattering and geophysics. As the solution of the Helmholtz equation oscillates severely for large wavenumbers, it is still a difficult computational problem to develop efficient numerical schemes to solve the Helmholtz equation at high wavenumbers.

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To numerically solve the Helmholtz equation, there are mainly finite difference methods (see [1]–[3]) and finite element methods (see [4]–[5]). The accuracy of finite element methods is higher than that of finite difference methods. However, the finite difference method is easily implemented, and its computational complexity is much less than that of the finite element method. Moreover, by optimizing the parameters in the finite difference formulas, we can minimize the numerical dispersion and improve the accuracy of the schemes (see [6]). In this paper, we consider the finite difference method for solving the Helmholtz equation with constant wavenumbers.

This paper is organized as follows. In Section 2, we construct a 3-point finite difference scheme for the 1D Helmholtz equation with constant wavenumbers. A convergence analysis is presented to show that the scheme is sixth-order in accuracy. To choose optimal weight parameters of the scheme, a refined choice strategy is also proposed. Numerical experiments are given to demonstrate the efficiency and accuracy of the scheme in Section 3. We show that the proposed scheme not only improves the accuracy but also reduces the numerical dispersion significantly. Finally, Section 4 contains the conclusions of this paper.

2 An Optimal Sixth-order Finite Difference Scheme for the Helmholtz Equation with Constant Wavenumbers

In this section, we propose a 3-point finite difference scheme for the 1D Helmholtz equation with constant wavenumbers. A convergence analysis is then provided to show that the scheme is sixth-order in accuracy. We also present a refined optimization rule for choosing the scheme's weight parameters based on minimizing the numerical dispersion.

2.1 A Sixth-order Finite Difference Scheme

We next present a 3-point finite difference method for the Helmholtz equation, and then prove that the proposed scheme is sixth-order in accuracy.

We begin with introducing some useful notations. To describe the finite difference scheme, we consider the network of grid points x_m, where $x_m := x_0 + (m-1)h$. Note that the same step size $h := \Delta x$ is used for the variable x. For each m, we set $u_m := u\,|_{x=x_m}$ and $k_m := k\,|_{x=x_m}$. Let $D_{xx}u$ denote the second-order centered-difference approximation for u_{xx}. We begin with establishing a sixth-order approximation for the term u_{xx}. By the Taylor expansion, we have

$$D_{xx}u = u_{xx} + \frac{h^2}{12}\frac{\mathrm{d}^4 u}{\mathrm{d}x^4} + \frac{h^4}{360}\frac{\mathrm{d}^6 u}{\mathrm{d}x^6} + \mathcal{O}(h^6). \tag{2.1}$$

To achieve an approximation of sixth-order with 3 points for u_{xx}, we need to construct an approximation of fourth-order for $\frac{\mathrm{d}^4 u}{\mathrm{d}x^4}$, and an approximation of second-order for $\frac{\mathrm{d}^6 u}{\mathrm{d}x^6}$ in the above equation. Moreover, both of the approximations should use only 3 points.

We rewrite $\frac{\mathrm{d}^4 u}{\mathrm{d}x^4}$ in the following proposition.