Communications in Mathematical Research
35(4)(2019), 335–339

Expanders, Group Extensions, Hadamard Manifolds and Certain Banach Spaces

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Communicated by Gong Gui-hua

Abstract: In this note, we prove that expanders cannot be coarsely embedded into group extensions of sequences of groups which are coarsely embeddable into Hardamad manifolds and certain Banach spaces due to the similar concentration theorems.

Key words: expander; group extension; Hadamard manifold; Banach space
2010 MR subject classification: 20E06; 46B99; 58C99
Document code: A
Article ID: 1674-5647(2019)04-335-05
DOI: 10.13447/j.1674-5647.2019.04.04

In [1], Arzhantseva and Tessera proved the property that expanders cannot be coarsely embedded into group extensions of sequences of groups which are coarsely embeddable into Hilbert spaces. In this note, we show that this property also holds for Hadamard manifolds and Banach spaces whose unit balls are uniformaly embeddable into Hilbert spaces.

First, let us recall basic definitions of expanders (see [2] for more information and for references). Let (V, E) be a finite graph with the vertex set V and the edge set E. Denote the cardinality of V and E by $|V| = n$ and $|E| = m$. We also define an orientation on E. The differential $d : \ell_2(V) \to \ell_2(E)$ is defined by

$$d(f) = f(e^+) - f(e^-)$$

for all $f \in \ell_2(V)$ and $e = (e^+, e^-) \in E$, where e^+ and e^- are initial and end points of e, respectively.

This differential $\boldsymbol{d}$ is an $m \times n$ matrix. The discrete Laplace operator $\Delta = d^*\boldsymbol{d}$, where d^* is the adjoint operator of d. This definition does not depend on the choice of the orientation of E. Δ is self-adjoint and positive. Hence it has real nonnegative eigenvalues. Denote by $\lambda_1(V)$ the minimal positive eigenvalue of the discrete Laplace operator Δ of the graph (V, E).

Received date: Feb. 20, 2019.
Foundation item: The NSF (11771117) of China.
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Definition 1 *A (k, λ) expander is a graph (V, E) with a fixed degree k and $\lambda_1(V) \geq \lambda$. A sequence of graphs (V_n, E_n) of a fixed degree k and with $|V_n| \to \infty$ is called an expanding sequence of graphs if there is a positive constant λ such that $\lambda_1(V_n) \geq \lambda$ for all $n \in \mathbf{N}$.*

Let X and Y be metric spaces and let $f : X \to Y$ be a map. Define the Lipschitz constant of f by

$$\mathrm{Lip}(f) = \sup\left\{\frac{d(f(s),\, f(t))}{d(s,\, t)} : s,\, t \in X \text{ and } s \neq t\right\}.$$

Lemma 1 *If X is a graph, then*

$$Lip(f) = \sup\{d(f(s),\, f(t)) : s,\, t \in X \text{ are adjacent}\}.$$

Proof. Let

$$\mathrm{Lip}_1(f) = \sup\{d(f(s),\, f(t)) : s,\, t \in X \text{ are adjacent}\}.$$

Clearly $\mathrm{Lip}_1(f) \leq \mathrm{Lip}(f)$. For any pair $s,\, t \in X$ with $d(s,\, t) = n$, there exists a sequence of points $x_0,\, x_1,\, \cdots,\, x_n$ of X such that $x_0 = s$, $x_n = t$ and $x_i,\, x_{i+1}$ are adjacent for all $i = 0, \cdots, n-1$. Then

$$\begin{aligned}\frac{d(f(s),\, f(t))}{d(s,\, t)} &= \frac{d(f(s),\, f(t))}{n} \\ &\leq \sum_{i=0}^{n-1} \frac{d(f(x_i),\, f(x_{i+1}))}{n} \\ &\leq n \cdot \frac{\mathrm{Lip}_1(f)}{n} \\ &= \mathrm{Lip}_1(f).\end{aligned}$$

Therefore, $\mathrm{Lip}(f) \leq \mathrm{Lip}_1(f)$. The proof is done.

Hence, for any pair $(s,\, t) \in G$, one always has

$$d(f(s),\, f(t)) \leq \mathrm{Lip}(f) d(s,\, t).$$

In [4], we have the following concentration theorem.

Theorem 1 [4] *Let M be a Hadamard manifold with bounded sectional curvatures. And let $(V,\, E)$ be a (k, λ) expander. Then there exists $R > 0$ such that for any $f : V \to M$,*

$$\frac{1}{|V|} \sum_{v \in V} d(f(v),\, m)^2 \leq R^2(\mathrm{Lip}(f) + 1)^2,$$

where m is a point such that $\sum\limits_{v \in V} \log_m(f(v)) = 0$.

In the following proposition, we see that if the average value is bounded by a number, then at least half of summands are bounded by the twice of the preceding number.

Proposition 1 *Let $a_1,\, \cdots,\, a_n$ be non-negative real numbers and $a > 0$. If*

$$\frac{1}{n} \sum_{i=1}^{n} a_i^2 \leq a^2,$$

then there are at least $\dfrac{n}{2}$ of $a_1,\, \cdots,\, a_n$ less than or equal to $2a$.