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Third Hankel Determinant for the Inverse of Starlike and Convex Functions

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Abstract: Denote $\mathcal{S}$ to be the class of functions which are analytic, normalized and univalent in the open unit disk $\mathbb{U} = \{z\colon |z| < 1\}$. The important subclasses of $\mathcal{S}$ are the class of starlike and convex functions, which we denote by $\mathcal{S}^*$ and $\mathcal{C}$. In this paper, we obtain the third Hankel determinant for the inverse of functions $f(z) = z + \sum\limits_{n=2}^{\infty} a_n z^n$ belonging to $\mathcal{S}^*$ and $\mathcal{C}$.

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1 Introduction

Let $\mathcal{H}(\mathbb{U})$ denote the class of functions which are analytic in the open unit disk $\mathbb{U} = \{z\colon |z| < 1\}$. Let $\mathcal{A}$ be the class of all functions $f \in \mathcal{H}(\mathbb{U})$ which are normalized by $f(0) = 0$ and $f'(0) = 1$ and have the following form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad z \in \mathbb{U}. \tag{1.1}$$

We denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of all functions in $\mathcal{A}$ which are also univalent in $\mathbb{U}$.

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In [1] and [2], the q-th Hankel determinant for $q \geqslant 1$ and $n \geqslant 1$ is stated by Pommerenke as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix},$$

where $n,\, q \in \mathbf{N}_+$.

Following Pommerenke, many authors focused on the investigating of the second Hankel determinant $H_2(2) = a_2a_4 - a_3^2$ (see [3]–[6]). Only a few papers have been devoted to the third Hankel determinant (see [7]–[11])

$$H_3(1) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix} = a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2). \tag{1.2}$$

We seek upper bound on the third Hankel determinant for the inverse of the classes $\mathcal{S}^*$ of starlike functions and $\mathcal{C}$ of convex functions. The class $\mathcal{S}^*$ and $\mathcal{C}$ are defined as follows.

Definition 1.1 *Let f be given by* (1.1). *Then $f \in \mathcal{S}^*$ if and only if*

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0, \qquad z \in \mathbb{U}. \tag{1.3}$$

Definition 1.2 *Let f be given by* (1.1). *Then $f \in \mathcal{C}$ if and only if*

$$\mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0, \qquad z \in \mathbb{U}. \tag{1.4}$$

Let $\mathcal{P}$ be the class of all function $p \in \mathcal{H}(\mathbb{U})$ satisfying $p(0) = 1$ and $\mathrm{Re}\{p(z)\} > 0$. The function $p \in \mathcal{P}$ have the following form:

$$p(z) = 1 + p_1z + p_2z^2 + p_3z^3 + \cdots, \qquad z \in \mathbb{U}. \tag{1.5}$$

In [7], it was proved that

Theorem 1.1

$$f \in \mathcal{S}^* \Rightarrow |H_3(1)| \leqslant 1,$$

$$f \in \mathcal{C} \Rightarrow |H_3(1)| \leqslant \frac{49}{540} = 0.090\cdots.$$

Lemma 1.1[12] *If $p \in \mathcal{P}$, then the sharp estimate $|p_n| \leqslant 2$ holds for $n = 1, 2, \cdots$.*

Lemma 1.2[13] *If $p \in \mathcal{P}$, then the following estimates holds for $n,\, k = 1, 2, \cdots,\ n > k$:*

$$|p_n - \lambda p_k p_{n-k}| \leqslant \begin{cases} 2, & 0 \leqslant \lambda \leqslant 1; \\ 2|2\lambda - 1|, & \lambda \geqslant 1. \end{cases}$$