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Univalent Criteria for Analytic Functions Involving Schwarzian Derivative

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Abstract: In this paper, some new criteria for univalence of analytic functions defined in the unit disk in terms of two parameters are presented. Moreover, the related result of Aharonov and Elias (Aharonov D, Elias U. Univalence criteria depending on parameters. *Anal. Math. Phys.*, 2014, **4**(1-2): 23–34) is generalized.

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1 Introduction

An analytic function $f(z)$ is said to be univalent in $\mathbb{D} = \{z, |z| < 1\}$ if it is one-to-one in $\mathbb{D}$. As usual, for some simple analytic functions we may judge easily if it is univalent by definition. In fact, we are often faced with complicated analytic function, and it is hard to determine whether to be univalent. Therefore, judging only by definition is not enough. This allows scholars to explore other univalent criteria. Recently, some new univalent criteria for analytic functions have been established in [1]–[4].

The Schwarzian derivative of a locally univalent analytic function $f(z)$ is defined by

$$S_{f(z)} = \left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2.$$

The status of $S_{f(z)}$ in the study of univalence is very important. Some classical univalent criteria over the Schwarzian derivative are introduced in the following. Nehari[5] proved that

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if

$$|S_{f(z)}| \leqslant \frac{2}{(1-|z|^2)^2}, \qquad z \in \mathbb{D} \tag{1.1}$$

or

$$|S_{f(z)}| \leqslant \frac{\pi^2}{2}, \qquad z \in \mathbb{D}, \tag{1.2}$$

then $f(z)$ is univalent in $\mathbb{D}$. Later, Hille[6] proved the criterion (1.1) is sharp. Pokornyi[7] stated the criterion

$$|S_{f(z)}| \leqslant \frac{4}{1-|z|^2}, \qquad z \in \mathbb{D}. \tag{1.3}$$

But this proof of the result is due to Nehari[8].

In the spirit of Steinmetz[9], Aharonov[10] defined a result of sharpness of univalent criteria.

Theorem 1.1 *A criterion for univalence of the form $|S_{f(z)}| \leqslant 2p(|z|)$ $(z \in \mathbb{D})$ is sharp, if for an analytic function $f(z)$, the conditions $S_{f(x)} \geqslant 2p(x)$, where $x \in (-1, 1)$, and $S_{f(x)} \neq 2p(x)$ imply that $f(z)$ is not univalent in $\mathbb{D}$.*

Nehari[8] proved the following theorem, which provides a method to establish new results on univalent criteria.

Theorem 1.2 *Suppose that*

(a) *$p(x)$ is a positive and continuous even function for $x \in (-1, 1)$;*

(b) *$p(x)(1-x^2)^2$ is nonincreasing for $x \in (0, 1)$;*

(c) *the differential equation*

$$y''(x) + p(x)y(x) = 0, \qquad x \in (-1, 1) \tag{1.4}$$

has a solution which does not vanish in $-1 < x < 1$. Then, any analytic function $f(z)$ in $\mathbb{D}$ satisfying $|S_{f(z)}| \leqslant 2p(|z|)$ is univalent in $\mathbb{D}$.

In view of Theorem 1.2, the univalent criteria (1.1), (1.2) and (1.3) can be given by

$$y(x) = \sqrt{1-x^2}, \qquad y(x) = \cos\frac{\pi x}{2}, \qquad y(x) = 1 - x^2,$$

respectively.

Let $p(z)$ be analytic in $\mathbb{D}$ and consider the analytic differential equation

$$y''(z) + p(z)y(z) = 0. \tag{1.5}$$

Further, let $u(z)$ and $v(z)$ be two linearly independent functions (solutions of (1.5)). Under the assumptions of Theorem 1.2, if $p(z)$ is self majorant (an analytic function $p(z)$ in the open $\mathbb{D}$ is said to be self majorant, if $|p(z)| \leqslant p(|z|)$ for all $z \in \mathbb{D}$), then

$$f_0(z) = \frac{v(z)}{u(z)}$$

satisfies

$$|S_{f_0(z)}| = 2|p(z)| \leqslant 2p(|z|),$$