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A Matrix Representation of Outer Derivations from $\mathfrak{gl}_{0|2}$ to the Generalized Witt Lie Superalgebra

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Abstract: Let $\mathfrak{gl}_{0|2}$ be a subalgebra of the general linear Lie superalgebra. In this paper, outer derivations from $\mathfrak{gl}_{0|2}$ to the generalized Witt Lie superalgebra are completely determined by matrices.

Key words: outer derivation; inner derivation; matrix representation; generalized Witt Lie superalgebra

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1 Introduction

The non-inner derivations are called outer derivations, which are in fact equal to the first cohomology groups of (super)algebras under consideration. For Lie (super)algebras, the existence of outer derivations have been well investigated (see [1]–[5] for examples). During the past half century, the theory of outer derivations and first cohomology groups for modular Lie (super)algebras has developed in a variety of directions and a large number of results have been obtained (see [6]–[8] for example). For classical Lie superalgebras over a field of prime characteristic, the recent papers [9] and [10] computed low-dimensional cohomology groups of the special linear Lie superalgebra $\mathfrak{sl}_{m|n}$ and its subalgebra $A(1;\,0)$ with coefficients in Witt or special superalgebras by virtue of the direct sum decomposition of submodules and the weight space decompositions of these submodules relative to their standard Cartan subalgebra.

The original motivation for this paper comes from [11]. The treatment of linear superal-

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gebras necessitates matrix computational techniques which are set forth in [12]. This paper contains a considerable amount of computation and determines the outer derivations from $\mathfrak{gl}_{0|2}$ to the generalized Witt Lie superalgebra W. Section 2 reviews the necessary notions. Section 3 calculates derivations and inner derivations from $\mathfrak{gl}_{0|2}$ into each irreducible submodule of W over a field of prime characteristic. Therefore, outer derivations from $\mathfrak{gl}_{0|2}$ to W are determined. In Section 4, the outer derivations from $\mathfrak{gl}_{0|2}$ to W over a field of characteristic zero are considered. Throughout this paper $\mathbb{F}$ is assumed to be an arbitrary field. The set of positive integers and the set of nonnegative integers are written as $\mathbf{N}_+$ and $\mathbf{N}$, respectively. Let L be a Lie algebra over $\mathbb{F}$ and A be an arbitrary L-module. Let $x \in L$, $a \in A$, we denote by $x \cdot a$ the element x acts on a.

2 Preliminaries

An $\mathbb{F}$-linear mapping $\varphi\colon L \to A$ is called a derivation from L into A if

$$\varphi([x, y]) = x \cdot \varphi(y) - y \cdot \varphi(x), \qquad x, y \in L. \tag{2.1}$$

Denote by $\mathrm{Der}(L, A)$ the derivation space from L into A. A derivation $\psi_a\colon L \to A$ is called inner if there is $a \in A$ such that

$$\psi_a(x) = x \cdot a, \qquad x \in L.$$

Denote by $\mathrm{Ider}(L, A)$ the inner derivation space from L into A. Denote by $\mathrm{Oder}(L, A)$ the outer derivation space from L into A. Thus

$$\mathrm{Oder}(L, A) = \mathrm{Der}(L, A)/\mathrm{Ider}(L, A).$$

This implies that the first cohomology group $\mathrm{H}^1(L, A)$ is isomorphic to $\mathrm{Oder}(L, A)$.

The following lemma is a standard fact. For more details see [13].

Lemma 2.1 *Suppose that A is an L-module and $A_1, A_2, \cdots, A_k$ are submodules of A such that $A = A_1 \bigoplus A_2 \bigoplus \cdots \bigoplus A_k$. Then*

$$\mathrm{H}^n(L, A) = \bigoplus_{i=1}^{k} \mathrm{H}^n(L, A_i), \qquad n \in \mathbf{N}.$$

According to Lemma 2.1, we obtain

$$\mathrm{Oder}(L, A) = \bigoplus_{i=1}^{k} \mathrm{Oder}(L, A_i), \tag{2.2}$$

which is frequently used in the sequel.

For sake of simplicity, let m, n denote fixed integers in $\mathbf{N}_+ \setminus \{1, 2\}$. For $\alpha = (\alpha_1, \cdots, \alpha_m) \in \mathbf{N}^m$, we put

$$|\alpha| := \sum_{i=1}^{m} \alpha_i.$$

Let $\mathcal{O}(m, \underline{t})$ denote the divided power algebra over $\mathbb{F}$ with an $\mathbb{F}$-basis $\{x^{(\alpha)} \mid \alpha \in \mathcal{O}(m, \underline{t})\}$, where

$$\mathcal{O}(m, \underline{t}) := \left\{\alpha := (\alpha_1, \cdots, \alpha_m) \in \mathbf{N}^m \mid 0 \le \alpha_i \leqslant p^{t_i} - 1,\ i = 1, 2, \cdots, m\right\}.$$

Let $\Lambda(n)$ be the exterior superalgebra over $\mathbb{F}$ in n variables $\xi_1, \xi_2, \cdots, \xi_n$ and $\mathcal{O}(m, n, \underline{t})$ denote the tenser product $\mathcal{O}(m, \underline{t}) \otimes_{\mathbb{F}} \Lambda(n)$.