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Remark on the Lifespan of Solutions to 3-D Anisotropic Navier Stokes Equations

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Abstract. The goal of this article is to provide a lower bound for the lifespan of smooth solutions to 3-D anisotropic incompressible Navier-Stokes system, which in particular extends a similar type of result for the classical 3-D incompressible Navier-Stokes system.

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1 Introduction

In this article, we shall investigate the lifespan for smooth enough solutions to the following 3-D anisotropic incompressible Navier-Stokes system:

$$\begin{cases} \partial_t u + u\cdot\nabla u - \Delta_{\mathrm{h}} u = -\nabla p, \\ \operatorname{div} u = 0, \\ u|_{t=0} = u_0, \end{cases} \tag{1.1}$$

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where $\Delta_{\rm h} \stackrel{\rm déf}{=} \partial_1^2+\partial_2^2$, u designates the velocity of the fluid and p the scalar pressure function which guarantees the divergence free condition of the velocity field.

Systems of this type appear in geophysical fluid dynamics (see for instance [5, 16]). In fact, meteorologists often modelize turbulent diffusion by putting a viscosity of the form: $-\mu_{\rm h}\Delta_{\rm h}-\mu_3\partial_3^2$, where $\mu_{\rm h}$ and μ_3 are empirical constants, and μ_3 is usually much smaller than $\mu_{\rm h}$. We refer to the book of Pedlovsky [16], Chap. 4 for a complete discussion about this model.

We remark that for the classical Navier-Stokes system (NS), which corresponds to (1.1) with $\Delta_{\rm h}$ there being replaced by $\Delta=\partial_1^2+\partial_2^2+\partial_3^2$, Leray [12] proved the global existence of weak solutions to (NS) in 1934. Yet the uniqueness and regularity to such weak solutions are still open. In [6], Chemin and Gallagher showed that: let u_0 be a regular solenoidal vector field, then the classical Navier-Stokes system (NS) has a unique regular solution on $[0,T]$. Let $T^\star(u_0)$ be the maximal time of existence of this regular solution. Then for any $\gamma\in(0,1/4)$, a positive constant C_γ exists so that

$$T^\star(u_0)\geq C_\gamma\|u_0\|_{\dot{H}^{\frac{1}{2}+2\gamma}}^{-\frac{1}{\gamma}}. \tag{1.2}$$

In the special case when $\gamma=\frac{1}{4}$, this type of result goes back to the seminal work of Leray [12]. Lately the same type of result has been proved for 3-D inhomogeneous incompressible Navier-Stokes system in [17] by the second author.

Considering that the system (1.1) has only horizontal dissipation, it is reasonable to use anisotropic Sobolev space defined as follows:

Definition 1.1. *For any (s,s') in $\mathbb{R}^2$, the anisotropic Sobolev space $\dot{H}^{s,s'}(\mathbb{R}^3)$ denotes the space of homogeneous tempered distribution a such that*

$$\|a\|_{\dot{H}^{s,s'}}^2 \stackrel{def}{=} \int_{\mathbb{R}^3}|\xi_{\rm h}|^{2s}|\xi_3|^{2s'}|\widehat{a}(\xi)|^2d\xi<\infty \quad with \quad \xi_{\rm h}=(\xi_1,\xi_2).$$

Mathematically, Chemin et al. [4] first studied the system (1.1). In particular, Chemin et al. [4] and Iftimie [11] proved that (1.1) is locally well-posed with initial data in $L^2\cap\dot{H}^{0,\frac{1}{2}+\varepsilon}$ for some $\varepsilon>0$, and is globally well-posed if in addition

$$\|u_0\|_{L^2}^{\varepsilon}\|u_0\|_{\dot{H}^{0,\frac{1}{2}+\varepsilon}}^{1-\varepsilon}\leq c \tag{1.3}$$

for some sufficiently small constant c.

Paicu [14] improved the well-posedness result in [4, 11] to be the critical case, namely, with initial data in the critical anisotropic Besov space, which basically