Commun. Math. Res.
doi: 10.4208/cmr.2020-0009

Vol. **36**, No. 2, pp. 154-192
June 2020

A Multiscale Multilevel Monte Carlo Method for Multiscale Elliptic PDEs with Random Coefficients

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Received 25 January 2020; Accepted 20 March 2020

Abstract. We propose a multiscale multilevel Monte Carlo (MsMLMC) method to solve multiscale elliptic PDEs with random coefficients in the multi-query setting. Our method consists of offline and online stages. In the offline stage, we construct a small number of reduced basis functions within each coarse grid block, which can then be used to approximate the multiscale finite element basis functions. In the online stage, we can obtain the multiscale finite element basis very efficiently on a coarse grid by using the pre-computed multiscale basis. The MsMLMC method can be applied to multiscale RPDE starting with a relatively coarse grid, without requiring the coarsest grid to resolve the smallest-scale of the solution. We have performed complexity analysis and shown that the MsMLMC offers considerable savings in solving multiscale elliptic PDEs with random coefficients. Moreover, we provide convergence analysis of the proposed method. Numerical results are presented to demonstrate the accuracy and efficiency of the proposed method for several multiscale stochastic problems without scale separation.

AMS subject classifications: 35J15, 65C05, 65N12, 65N30, 65Y20

Key words: Random partial differential equations (RPDEs), uncertainty quantification (UQ), multiscale finite element method (MsFEM), multilevel Monte Carlo (MLMC), reduced basis, convergence analysis.

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1 Introduction

Many physical and engineering applications involving uncertainty quantification (UQ) can be described by stochastic partial differential equations (SPDEs, i.e., PDEs driven by Brownian motion) or partial differential equations with random coefficients (RPDEs). In recent years, there has been an increased interest in the simulation of systems with uncertainties, and several numerical methods have been developed in the literature to solve SPDEs and RPDEs; see e.g. [2, 14, 15, 21, 25, 27, 28, 31, 33, 35, 37, 38]. These methods can be effective when the dimension of stochastic input variables is low. However, their performance deteriorates dramatically when the dimension of stochastic input variables is high because of the curse of dimensionality.

There are some attempts in developing sparsity or data-driven basis to attack these challenging problems. Most of them take advantage of the fact that even though the stochastic input has high dimension, the solution actually lives in a relatively low dimensional space. Therefore, one can develop certain sparsity or data-driven basis functions to solve the SPDEs and RPDEs efficiently. In [7–9,20,39,40], Hou et al. explored the Karhunen-Loève expansion of the stochastic solution, and constructed problem-dependent stochastic basis functions to solve these SPDEs and RPDEs. In [11, 26], the compressive sensing technique is employed to identify a sparse representation of the solution in the stochastic direction. In [5, 6], Schwab et al. studied the sparse tensor discretization of elliptic RPDEs.

In this paper, we consider another challenge in UQ, i.e., solving multiscale elliptic PDEs with random coefficients. Due to the large range of scales in these solutions, it requires tremendous computational resources to resolve the small scales of the solution. We propose a multiscale multilevel Monte Carlo method (MsMLMC) to significantly reduce the computational cost in solving multiscale elliptic PDEs with random coefficients. We use the following elliptic equation with multiscale random coefficient as an example to illustrate the main idea of our approach:

$$-\nabla\cdot\left(a^{\varepsilon}(x,\omega)\nabla u^{\varepsilon}(x,\omega)\right)=f(x),\quad x\in D,\ \omega\in\Omega, \tag{1.1}$$

$$u^{\varepsilon}(x,\omega)=0,\quad x\in\partial D, \tag{1.2}$$

where $D\subset R^d$ is a bounded spatial domain, Ω is a sample space, and $f(x)\in L^2(D)$. The multiscale information is described by the multiscale coefficient $a^{\varepsilon}(x,\omega)$. The precise definition of the $a^{\varepsilon}(x,\omega)$ will be given in Section 3.1.

Our MsMLMC method consists of two steps. In the first step, we apply the non-intrusive method (Monte Carlo or stochastic collocation method) to dis-