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A Comparison Study of Deep Galerkin Method and Deep Ritz Method for Elliptic Problems with Different Boundary Conditions

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Abstract. Recent years have witnessed growing interests in solving partial differential equations by deep neural networks, especially in the high-dimensional case. Unlike classical numerical methods, such as finite difference method and finite element method, the enforcement of boundary conditions in deep neural networks is highly nontrivial. One general strategy is to use the penalty method. In the work, we conduct a comparison study for elliptic problems with four different boundary conditions, i.e., Dirichlet, Neumann, Robin, and periodic boundary conditions, using two representative methods: deep Galerkin method and deep Ritz method. In the former, the PDE residual is minimized in the least-squares sense while the corresponding variational problem is minimized in the latter. Therefore, it is reasonably expected that deep Galerkin method works better for smooth solutions while deep Ritz method works better for low-regularity solutions. However, by a number of examples, we observe that deep Ritz method can outperform deep Galerkin method with a clear dependence of dimensionality even for smooth solutions and deep Galerkin method can also outperform deep Ritz method for low-regularity solutions. Besides, in some cases, when the boundary condition can be implemented in an exact manner, we find that such a strategy not only provides a better approximate solution but also facilitates the training process.

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1 Introduction

In the past decade, deep learning has achieved great success in many subjects, like computer vision, speech recognition, and natural language processing [7, 11, 18] due to the strong representability of deep neural networks (DNNs). Meanwhile, DNNs have also been used to solve partial differential equations (PDEs); see for example [1, 4, 5, 8, 19, 22, 24, 26, 28]. In classical numerical methods such as finite difference method [20] and finite element method [2], the number of degrees of freedoms (dofs) grows exponentially fast as the dimension of PDE increases. One striking advantage of DNNs over classical numerical methods is that the number of dofs only grows (at most) polynomially. Therefore, DNNs are particularly suitable for solving high-dimensional PDEs. The magic underlying this is to approximate a function using the network representation of independent variables without using mesh points. Afterwards, Monte-Carlo method is used to approximate the loss (objective) function which is defined over a high-dimensional space. Some methods are based on the PDE itself [24, 26] and some other methods are based on the variational or the weak formulation [5,16,21,28]. Another successful example is the multilevel Picard approximation which is provable to overcome the curse of dimensionality for a class of semilinear parabolic equations [13]. In the current work, we focus on two representative methods: deep Ritz method (DRM) proposed by E and Yu [5] and deep Galerkin method (DGM) proposed by Sirignano and Spiliopoulos [26]. It is worth mentioning that the loss function in DGM is defined as the PDE residual in the least-squares sense. Therefore, DGM is not a Galerkin method and has no connection with Galerkin from the perspective of numerical PDEs although it is named after Galerkin.

In classical numerical methods, boundary conditions can be exactly enforced for mesh points at the boundary. Typically boundary conditions include Dirichlet, Neumann, Robin, and periodic boundary conditions [6]. However, it is very difficult to impose exact boundary conditions for a DNN representation. Therefore, in the loss function, it is often to add a penalty term which penalizes the difference between the DNN representation on the boundary and the exact boundary condition, typically in the sense of L^2 norm. Only when Dirichlet boundary condition is imposed, a novel construction of two DNN representations can be