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Approximate Controllability of the System Governed by Double Coupled Semilinear Degenerate Parabolic Equations

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Abstract. This paper concerns the approximate controllability of the initial-boundary problem of double coupled semilinear degenerate parabolic equations. The equations are degenerate at the boundary, and the control function acts in the interior of the spacial domain and acts only on one equation. We overcome the difficulty of the degeneracy of the equations to show that the problem is approximately controllable in L^2 by means of a fixed point theorem and some compact estimates. That is to say, for any initial and desired data in L^2, one can find a control function in L^2 such that the weak solution to the problem approximately reaches the desired data in L^2 at the terminal time.

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Key words: Double coupled, degeneracy, approximate controllability.

1 Introduction

In this paper, we investigate the approximate controllability of the following semilinear problem:

$$\frac{\partial u}{\partial t}-\operatorname{div}(a_1(x,t)\nabla u)+g(x,t,u,v)=h(x,t)\chi_{\omega_1},\quad (x,t)\in Q_T, \tag{1.1}$$

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$$\begin{aligned}
&\frac{\partial v}{\partial t}-\operatorname{div}(a_2(x,t)\nabla v)+q(x,t,v)=u\chi_{\omega_2}, && (x,t)\in Q_T, && (1.2)\\
&u(x,t)=0, && (x,t)\in\Sigma_1, && (1.3)\\
&v(x,t)=0, && (x,t)\in\Sigma_2, && (1.4)\\
&u(x,0)=u_0(x), && x\in\Omega, && (1.5)\\
&v(x,0)=v_0(x), && x\in\Omega, && (1.6)
\end{aligned}$$

where $Q_T=\Omega\times(0,T)$, Ω is a bounded smooth domain in $\mathbb{R}^n$, $T>0$, $h\in L^2(Q_T)$ is the control function, χ is the characteristic function, ω_1 and ω_2 are open subsets of Ω satisfying $\omega_1\cap\omega_2\neq\emptyset$, $a_1,a_2\in C(\overline{Q}_T)$ satisfy

$$a_i(x,t)>0 \quad \text{for} \quad (x,t)\in Q_T, \quad \frac{1}{a_i}\frac{\partial a_i}{\partial t}\in L^\infty(Q_T), \quad i=1,2,$$

$u_0,v_0\in L^2(\Omega)$, g and q are measurable functions in $Q_T\times\mathbb{R}\times\mathbb{R}$ and $Q_T\times\mathbb{R}$, respectively. Furthermore, g and q satisfy

$$\begin{aligned}
&g(x,t,\cdot,\cdot)\in C^1(\mathbb{R}\times\mathbb{R}) \quad \text{uniformly for} \quad (x,t)\in Q_T, && (1.7)\\
&\left|\frac{\partial g}{\partial s}(x,t,s,p)\right|+\left|\frac{\partial g}{\partial p}(x,t,s,p)\right|\leq M, \qquad (x,t,s,p)\in Q_T\times\mathbb{R}\times\mathbb{R}, && (1.8)\\
&|q(x,t,u)-q(x,t,v)|\leq M|u-v|, \qquad (x,t)\in Q_T, \quad u,v\in\mathbb{R}, && (1.9)
\end{aligned}$$

where $M>0$ is a constant. Since (1.1) and (1.2) may be degenerate at the lateral boundary, the boundary conditions are prescribed not on the whole lateral boundary, but only on Σ_1 and Σ_2, which are the nondegenerate and weakly degenerate parts of the lateral boundary. More precisely,

$$\begin{aligned}
\Sigma_i=\Big\{(x,t)\in\partial\Omega\times(0,T):a_i(x,t)>0\Big\}\cup\Big\{&(x,t)\in\partial\Omega\times(0,T):a_i(x,t)=0\\
&\text{and there exists} \quad 0<\delta<\min\{t,T-t\} \quad \text{such that}\\
&\int_{t-\delta}^{t+\delta}\int_{\{y\in\Omega:|y-x|<\delta\}}\frac{1}{a_i(y,s)}\mathrm{d}y\mathrm{d}s<+\infty\Big\}, \quad i=1,2.
\end{aligned}$$

The degenerate equations (1.1) and (1.2) can be used to describe some models from mathematical biology and physics, such as the Lotka-Volterra model and the Keller-Legel model [2,17]. The degenerate equations (1.1) and (1.2) are double coupled, whose linearized equations are of the form

$$\begin{aligned}
&\frac{\partial u}{\partial t}-\operatorname{div}(a_1(x,t)\nabla u)+c_1(x,t)u+c_2(x,t)v=h(x,t)\chi_{\omega_1}, && (x,t)\in Q_T, && (1.10)\\
&\frac{\partial v}{\partial t}-\operatorname{div}(a_2(x,t)\nabla v)+c_3(x,t)v=u\chi_{\omega_2}, && (x,t)\in Q_T, && (1.11)
\end{aligned}$$