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Bounding Topology via Geometry, A-Simple Fundamental Groups

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Abstract. We call a group A-simple, if it has no non-trivial normal abelian subgroup. We will present finiteness results in controlled topology via geometry on manifolds whose fundamental groups are A-simple.

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Key words: A-simple fundamental group, collapsing with bounded sectional curvature, finiteness of fundamental groups and diffeomorphic types.

1 Introduction

This paper may be treated as a continuation of [25], where we investigated some problems of controlled topology via geometry. In this paper, we are still confined to the class of compact manifolds with bounded sectional curvature and diameter.

Let us start with the classical Cheeger diffeomorphism finiteness theorem [3].

Theorem 1.1 (Diffeomorphism finiteness of noncollapsed manifolds)**.** *Given $n,d,v>0$, there exists a constant $C(n,d,v)>0$ such that the collection of compact n-manifolds satisfying*

$$|\mathrm{sec}_M|\le 1,\quad \mathrm{diam}(M)\le d,\quad \mathrm{vol}(M)\ge v,$$

contains at most $C(n,d,v)$ many diffeomorphic types.

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A different type of diffeomorphism finiteness result in Riemannian geometry, which primarily concerns collapsed manifolds with bounded sectional curvature and diameter, was obtained independently in [8, 10] and [21].

Theorem 1.2 (π_2-diffeomorphism finiteness of collapsed manifolds)**.** *Given $n,d>0$, there exists a constant $C(n,d)>0$ such that the collection of compact n-manifolds satisfying*

$$|\sec_M|\le 1,\quad \mathrm{diam}(M)\le d,\quad \pi_1(M)=\pi_2(M)=0,$$

contains at most $C(n,d)$ many diffeomorphic types.

The Berger's collapsed 3-sphere satisfies Theorem 1.2. A proof of Theorem 1.2 relies on the singular nilpotent fibration theorem (see [4, 12, 13, Theorem 1.4]).

Note that for $2\le n\le 6$, Theorem 1.2 remains true without the condition that $\pi_2(M)=0$ [9]. In [25], we generalized Theorem 1.2 to the following:

Theorem 1.3 (π_2-diffeomorphism finiteness on universal covers)**.** *Given $n,d>0$, there exists a constant $C(n,d)>0$ such that the collection of the universal covers of compact n-manifolds satisfying*

$$|\sec_M|\le 1,\quad \mathrm{diam}(M)\le d,\quad \pi_2(M)=0,$$

contains at most $C(n,d)$ many diffeomorphic types.

Restricting to simply connected manifolds, Theorem 1.3 reduces to Theorem 1.2, while the Riemannian universal covers in Theorem 1.3 may not be compact.

In view of the above diffeomorphism finiteness results, the following questions are natural:

Problem 1.1. If one replaces the condition $\mathrm{vol}(M)\ge v$ in Theorem 1.1 by $\mathrm{vol}(B_1(\tilde p))\ge v$, that is, $\mathrm{vol}(M)$ may be very small, but $\mathrm{vol}(B_1(\tilde p))\ge v>0$, where $\tilde p$ is a point in the Riemannian universal cover $\tilde M$ of M (e.g., $S^3/\mathbb{Z}_q\times\varepsilon S^1$ with $q\in\mathbb{Z}$ fixed and ε is a small number), what topological constraint on M one may have?

Problem 1.2. ([10]) If one replaces the condition $\pi_1(M)=1$ in Theorem 1.2 by a class of $\pi_1(M)$ including $\pi_1(M)=1$, does the diffeomorphism finiteness still hold for M (instead of $\tilde M$, see Theorem 1.3)?

The main efforts in this paper is to provide some answers to Problems 1.1 and 1.2, which are closely related to a property of $\pi_1(M)$. We say that a finitely presented group Γ is A-simple, if Γ contains only trivial normal abelian subgroup