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Global Solutions of Modified One-Dimensional Schrödinger Equation

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Abstract. In this paper, we consider the modified one-dimensional Schrödinger equation:

$$\big(D_t - F(D)\big)u = \lambda|u|^2 u,$$

where $F(\xi)$ is a second order constant coefficients classical elliptic symbol, and with smooth initial datum of size $\varepsilon \ll 1$. We prove that the solution is global-in-time, combining the vector fields method and a semiclassical analysis method introduced by Delort. Moreover, we get a one term asymptotic expansion for u when $t \to +\infty$.

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Key words: Schrödinger equation, semiclassical Analysis, global solution.

1 Introduction

We consider the following modified one-dimensional Schrödinger equation:

$$\begin{cases} \big(D_t - F(D)\big)u = \lambda|u|^2 u, \quad t>0, \quad x \in \mathbb{R}, \\ u(x,0) = \varepsilon u_0(x), \end{cases} \tag{1.1}$$

where $D_t = \partial_t / i$, $D = \partial_x / i$, $F(\xi)$ is a second order constant coefficients classical elliptic symbol, u is a complex valued function, and $\lambda = 1$ or $\lambda = -1$ correspons

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to the defocusing or the focusing case. We assume that $F(\xi)$ is a smooth function defined on $\mathbb{R}$, $\xi \to F(\xi) \in \mathbb{R}$, satisfying

$$F(\xi) \in C^\infty(\mathbb{R}), \quad |F(\xi)| \le c_0 \left(1+|\xi|^2\right), \quad 0 < c_1 \le F''(\xi) \le c_2, \quad \text{for all} \quad \xi \in \mathbb{R} \tag{1.2}$$

for some positive constants c_i, $i=0,1,2$. For example, we can choose a smooth function $F(\xi)$, which has an expansion

$$F(\xi) = c_\pm^2 \xi^2 + c_\pm^1 \xi + c_\pm^0 + c_\pm^{-1} \xi^{-1} + c_\pm^{-2} \xi^{-2} + \dots,$$

when ξ goes to $\pm\infty$, where $c_\pm^2 > 0$.

For the classical one-dimensional Schrödinger equation

$$\begin{cases} iu_t + \dfrac{1}{2} u_{xx} = \lambda |u|^2 u, \quad t>0, \quad x \in \mathbb{R}, \\ u(x,0) = \varepsilon u_0(x) \end{cases} \tag{1.3}$$

there are many papers that studied the global well-posedness problem, decay and the asymptotic behavior of the solution, see [4–6,9,11,13,16,20]. It has a modified linear scattering

$$u(x,t) = \frac{1}{\sqrt{t}} e^{\frac{ix^2}{2t}} W\left(\frac{x}{t}\right) e^{-i\lambda \ln t |W(\frac{x}{t})|^2} + err_x, \tag{1.4}$$

$$\widehat{u}(\xi,t) = e^{-\frac{it\xi^2}{2}} W(\xi) e^{-i\lambda \ln t |W(\xi)|^2} + err_\xi, \tag{1.5}$$

where

$$\begin{aligned} err_x &= \varepsilon O_{L_x^\infty}\left((1+t)^{-\frac{3}{4}+C\varepsilon^2}\right) \cap O_{L_x^2}\left((1+t)^{-1+C\varepsilon^2}\right), \\ err_\xi &= \varepsilon O_{L_\xi^\infty}\left((1+t)^{-\frac{1}{4}+C\varepsilon^2}\right) \cap O_{L_\xi^2}\left((1+t)^{-\frac{1}{2}+C\varepsilon^2}\right). \end{aligned}$$

For the classical one-dimensional Schrödinger equation (1.3), there is a vector field $L = x + it\partial_x$, which is the generator of the Galilean group of symmetries, satisfying

$$e^{\frac{it}{2}\partial_x^2} x = L e^{\frac{it}{2}\partial_x^2}, \quad \left[i\partial_t + \frac{1}{2}\partial_x^2, L \right] = 0,$$

and

$$L\left(u|u|^2\right) = 2|u|^2 Lu - u^2 \overline{Lu}. \tag{1.6}$$

For the modified Schrödinger equation (1.1), we also can define a vector field

$$\mathcal{L} = x + tF'(D), \tag{1.7}$$