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A Finite Method for Computing the Drazin and Core-EP Inverses of Matrices Based on Partial Full-Rank Factorization

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Abstract. This paper presents a formula for the Drazin inverses of matrices based on a sequence of partial full-rank factorizations which theoretically extends the classic full-rank factorization method for computing the Drazin inverses established by R.E. Cline. The result is then extended to the core-EP inverses.

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1 Introduction

Throughout this paper we shall use the standard notation: $R(A)$, $\text{rank}(A)$, and A^* for the range, the rank, and the conjugate transpose of A [2,34]. Let $A \in \mathbb{C}^{m\times n}$, the Moore-Penrose inverse $A^\dagger$ of A is the unique matrix $X \in \mathbb{C}^{n\times m}$ that satisfies the following four matrix equations [26]:

$$AXA = A, \quad XAX = X, \quad (AX)^* = AX, \quad (XA)^* = XA.$$

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The Drazin inverse A^D of a square matrix $A \in \mathbb{C}^{n\times n}$ is the unique matrix $X \in \mathbb{C}^{n\times n}$ that satisfies the following three matrix equations [8]:

$$XAX = X, \quad AX = XA, \quad A^{k+1}X = A^k,$$

where $k = \text{Ind}(A)$ is the index of A, which is defined as the smallest nonnegative integer m such that rank(A^{m+1}) = rank(A^m). Recently, there has been growing interest in the study of a generalized inverse known as core-EP inverse $A^{\textcircled{\dagger}}$. The core-EP inverse $A^{\textcircled{\dagger}}$ of $A \in \mathbb{C}^{n\times n}$ is the unique matrix $X \in \mathbb{C}^{n\times n}$ that satisfies [27]

$$XAX = X, \quad R(X) = R(X^*) = R(A^k),$$

where $k = \text{Ind}(A)$. It is easily seen that, when A is invertible, the Moore-Penrose, the Drazin, and the core-EP inverses of A all collapse to the regular inverse A^{-1}. Therefore, throughout this paper we assume that A is singular.

There exist many expressions and interesting properties for the core-EP inverse $A^{\textcircled{\dagger}}$ of A and its extensions in [9, 10, 18, 20–22, 24, 25, 27, 28, 32, 35] and the references therein. Let us recall one of these expressions

$$A^{\textcircled{\dagger}} = A^D A^m (A^m)^\dagger \quad \text{for} \quad m \geq \text{Ind}(A). \tag{1.1}$$

In view of (1.1), the core-EP inverse is a combination of the Drazin and the Moore-Penrose inverses. It is recently observed that the solution of $\min_{x\in R(A^k)} \|b - Ax\|_2$, where $k = \text{Ind}(A)$ and $\|\cdot\|_2$ is the 2-norm, can be expressed as $A^{\textcircled{\dagger}}b$ [18,24] and its generalization to quaternion matrices with arbitrary index can be found in [19].

The generalized inverses of matrices are extremely useful in applied mathematics and some recent applications of generalized inverses can be found, for instance, in [13–15, 23, 29]. The computation of generalized inverses has been the primary focus recently, from finite methods such as Greville and Gauss elimination algorithms to Newton types of iterative schemes. In particular, many finite algorithms have been investigated by several authors in the literature. Those in [3,4,11,31,33] are based on the computation of the generalized inverse of rank-one updates and the ones in [12, 16, 17, 30] are through elementary row operations. Other finite algorithms are achieved by reducing the computation of the generalized inverses of A into one or two problems of smaller scale. One of such algorithms is the divide-and-conquer algorithm [5].

There are also quite a few methods for the computation of Drazin inverses [1, 2, 34]. In particular, the full-rank factorization method for the Drazin inverse established by Cline [6, 7] stands out among many finite methods due to its simplicity and the fact that it does not require any information of the index of A.