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The Pseudo Drazin inverses in Banach Algebras

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Dedicate to Sir Roger Penrose for his 90th birthday with our deepest admiration.

Abstract. Let $\mathscr{A}$ be a complex Banach algebra and J be the Jacobson radical of $\mathscr{A}$. (1) We firstly show that a is generalized Drazin invertible in $\mathscr{A}$ if and only if $a+J$ is generalized Drazin invertible in $\mathscr{A}/J$. Then we prove that a is pseudo Drazin invertible in $\mathscr{A}$ if and only if $a+J$ is Drazin invertible in $\mathscr{A}/J$. As its application, the pseudo Drazin invertibility of elements in a Banach algebra is explored. (2) The pseudo Drazin order is introduced in $\mathscr{A}$. We give the necessary and sufficient conditions under which elements in $\mathscr{A}$ have pseudo Drazin order, then we prove that the pseudo Drazin order is a pre-order.

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Key words: Drazin inverse, pseudo Drazin inverse, generalized Drazin inverse.

1 Introduction

Throughout this paper, $\mathscr{A}$ is a complex Banach algebra with unity $1_{\mathscr{A}}$. The symbols $J=J(\mathscr{A})$, $\mathscr{A}^{-1}$ denote the Jacobson radical of $\mathscr{A}$, the set of all invertible elements of A. If $a\in\mathscr{A}$, then the spectrum of a is $\sigma(a)=\{\lambda\in\mathbb{C}:\lambda 1_{\mathscr{A}}-a \text{ is not invertible}\}$.

In 1958, Drazin [5] introduced the concept of pseudo-inverse, which is called Drazin inverse later. An element $a\in\mathscr{A}$ is called Drazin invertible if there exists

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$b\in\mathscr{A}$ satisfying the following equations:

(i) $ab=ba$,

(ii) $ab^2=b$,

(iii) $a^k=a^{k+1}b$ for some integer $k\geq 1$.

If such an element b exists, it is unique and called the Drazin inverse of a (denoted by a^D). The smallest positive integer k satisfying equations (i)-(iii) is called the Drazin index of a and denoted by $i(a)$. The set of all Drazin invertible elements in $\mathscr{A}$ is denoted by $\mathscr{A}^D$.

In 1996, Koliha [7] introduced the concept of generalized Drazin inverses in Banach algebras. An element $a\in\mathscr{A}$ is called generalized Drazin invertible if there exists $b\in\mathscr{A}$ satisfying the following equations:

(I) $ab=ba$,

(II) $ab^2=b$,

(III) $a-a^2b\in\mathscr{A}^{\rm qnil}$,

where $\mathscr{A}^{\rm qnil}=\{x\in\mathscr{A}:\sigma(x)=0\}$. If such an element b exists, it is unique and called the generalized Drazin inverse of a (denoted by a^{gD}). The set of all generalized Drazin invertible elements in $\mathscr{A}$ is denoted by $\mathscr{A}^{gD}$.

In 2012, Wang and Chen [14] introduced the concept of pseudo Drazin inverses in rings and Banach algebras. An element $a\in\mathscr{A}$ is called pseudo Drazin invertible if there exists $b\in\mathscr{A}$ satisfying the following equations:

(1) $ab=ba$,

(2) $ab^2=b$,

(3) $a^k-a^{k+1}b\in J$ for some integer $k\geq 1$.

If such an element b exists, it is unique and called the pseudo Drazin inverse of a (denoted by $a^{\ddagger}$). The smallest positive integer k satisfying conditions (1)-(3) is called the pseudo Drazin index of a and denoted by $I(a)$. The set of all pseudo Drazin invertible elements in $\mathscr{A}$ is denoted by $\mathscr{A}^{pD}$. They pointed out that

$$\mathscr{A}^D\subseteq\mathscr{A}^{pD}\subseteq\mathscr{A}^{gD}.$$

Since then, a lot of results on the generalized Drazin inverses and pseudo Drazin inverses have been achieved, see [1,2,4,8,9,12,13,15–18].