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Perturbation of the Weighted T-Core-EP Inverse of Tensors Based on the T-Product

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Abstract. In this paper, we extend the notion of the T-Schur decomposition to the weighted T-core-EP decomposition. Next, the weighted T-core-EP inverse of rectangular tensors is defined by a system, and its existence and uniqueness are obtained. Furthermore, the perturbation of the weighted T-core-EP inverse is studied under several conditions, and the relevant examples are provided to verify the perturbation bounds of the weighted T-core-EP inverse of tensors.

AMS subject classifications: 15A69, 65F10

Key words: T-product, tensor, perturbation analysis, weighted T-core-EP inverse.

1 Introduction

Tensors are geometric objects that represent the concept of higher order array. In [29], Qi defined the symmetric hyperdeterminant, eigenvalues and eigenvectors of a real symmetric tensor, and found that their structures have a close link with the positive definiteness issue. After that, mathematical modelling and methodology based on tensors have made progresses in various fields [2, 4, 5, 7, 17, 35]. There are many works on tensors in recent years [6, 30, 31, 38]. In particular, one-order tensors and two-order tensors are vectors and matrices, respectively. This paper mainly investigates the weighted T-core-EP inverse of third-order tensors. For positive integers m,n and p, the set of all $m\times n\times p$ complex tensors and real

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tensors are denoted by $\mathbb{C}^{m\times n\times p}$ and $\mathbb{R}^{m\times n\times p}$. Denote $\mathcal{O}$ as a zero tensor while all the entries of tensor are zeros. It is well known that the Einstein product is a multiplication operation between tensors. There are several generalized inverses of tensors have been presented based on the Einstein product [1,13,21,32,34]. In [15], a new type of product between third-order tensors called the T-product is provided by Kilmer, which offers a new contribution to the class of tensors based algorithms for compression. Third-order tensors could be used in variety of application areas [16,26,33]. Relevant researches present the generalized inverses of tensors based on the T-product. In [23], Miao introduced the T-Moore-Penrose inverse and the T-Drazin inverse of tensors based on the T-product. Let $\mathcal{A}\in\mathbb{C}^{m\times n\times p}$, $\mathcal{A}^H\in\mathbb{C}^{n\times m\times p}$ be the conjugate transpose of $\mathcal{A}$, then the unique tensor $\mathcal{X}\in\mathbb{C}^{n\times m\times p}$ which satisfying

$$\mathcal{A}*\mathcal{X}*\mathcal{A}=\mathcal{A},\quad \mathcal{X}*\mathcal{A}*\mathcal{X}=\mathcal{X},\quad (\mathcal{A}*\mathcal{X})^H=\mathcal{A}*\mathcal{X},\quad (\mathcal{X}*\mathcal{A})^H=\mathcal{X}*\mathcal{A}$$

is called the T-Moore-Penrose inverse of $\mathcal{A}$ and denoted by $\mathcal{A}^\dagger$. Let $\mathrm{Ind}_T(\mathcal{A})$ represent the T-index of a frontal square tensor $\mathcal{A}$, and the T-rank of $\mathcal{A}$ is denoted by $\mathrm{rank}_T(\mathcal{A})$, then it is obviously that $\mathrm{Ind}_T(\mathcal{A})=k$ is the smallest positive integer such that $\mathrm{rank}_T(\mathcal{A}^k)=\mathrm{rank}_T(\mathcal{A}^{k+1})$. If $\mathcal{A}\in\mathbb{C}^{n\times n\times p}$, the T-Drazin inverse $\mathcal{A}^D\in\mathbb{C}^{n\times n\times p}$ is defined by

$$\mathcal{A}^k*\mathcal{A}^D*\mathcal{A}=\mathcal{A}^k,\quad \mathcal{A}^D*\mathcal{A}*\mathcal{A}^D=\mathcal{A}^D,\quad \mathcal{A}*\mathcal{A}^D=\mathcal{A}^D*\mathcal{A},$$

where k is the T-index of $\mathcal{A}$. In [39], for a tensor $\mathcal{A}\in\mathbb{C}^{n\times n\times p}$, the T-core inverse of $\mathcal{A}$ is defined by

$$\mathcal{A}*\mathcal{X}=\mathcal{P}_{\mathcal{A}},\quad \mathcal{R}(\mathcal{X})\subseteq\mathcal{R}(\mathcal{A}),$$

the tensor $\mathcal{X}$ is called the T-core inverse of $\mathcal{A}$ and denoted by $\mathcal{A}^{\text{#}}$, where $\mathcal{R}(\mathcal{A})$ is the range space of $\mathcal{A}$ and $\mathcal{P}_{\mathcal{A}}$ is the orthogonal projector onto $\mathcal{R}(\mathcal{A})$. Furthermore, $\mathcal{A}$ is T-core invertible if and only if $\mathrm{Ind}_T(\mathcal{A})\le 1$. In [8], the unique tensor $\mathcal{X}\in\mathbb{C}^{n\times n\times p}$ such that

$$\mathcal{X}*\mathcal{A}*\mathcal{X}=\mathcal{X},\quad \mathcal{R}(\mathcal{X})=\mathcal{R}(\mathcal{X}^H)=\mathcal{R}(\mathcal{A}^k)$$

holds, then $\mathcal{X}$ is called the T-core-EP inverse of $\mathcal{A}$ and denoted by $\mathcal{A}^{\oplus}$.

In matrix analysis, for a square matrix with index k, it can be expressed as the product of a unitary matrix and a upper triangular matrix by the Schur decomposition, and the expression of the core-EP inverse of matrices under the Schur decomposition has been carried out in [28]. There are several studies concerning on the characterizations and perturbations of the core-EP inverse of matrices, which can be referred in [11,20,37]. In [9], Ferreyra extended the Schur decomposition of square matrices to the weighted core-EP decomposition of rectangular