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Further Results on the Drazin Inverse of Tensors via Einstein Product

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Abstract. In this paper, we give further results on the Drazin inverse of tensors via the Einstein product. We give a limit formula for the Drazin inverse of tensors. By using this formula, the representations for the Drazin inverse of several block tensor are obtained. Further, we give the Drazin inverse of the sum of two tensors based on the representation for the Drazin inverse of a block tensor.

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1 Introduction

For a positive integer N, let $[N]=\{1,\dots,N\}$. An order N tensor $\mathcal{A}=(a_{i_1\dots i_N})_{1\leqslant i_j\leqslant I_j}$ $(j\in[N])$ is a multidimensional array with $I_1I_2\cdots I_N$ entries [15]. Clearly, an order 2 tensor is a matrix. Let $\mathbb{C}^{I_1\times\cdots\times I_N}$ denote the set of all $I_1\times\cdots\times I_N$ dimension order N tensors over complex field. A tensor $\mathcal{A}=(a_{i_1\cdots i_N j_1\cdots j_N})\in\mathbb{C}^{I_1\times\cdots\times I_N\times I_1\times\cdots\times I_N}$ whose entries $a_{i_1\cdots i_N i_1\cdots i_N}=1$ $(i_j\in[I_j], j\in[N])$ and the other entries are zero is a unit tensor, denoted by $\mathcal{I}$.

For tensors $\mathcal{A}\in\mathbb{C}^{I_1\times\cdots\times I_{N_1}\times K_1\times\cdots\times K_N}$ and $\mathcal{B}\in\mathbb{C}^{K_1\times\cdots\times K_N\times J_1\times\cdots\times J_{N_2}}$, the Einstein product of $\mathcal{A}$ and $\mathcal{B}$, denoted by $\mathcal{A}*_N\mathcal{B}$, is an order N_1+N_2 dimension $I_1\times\cdots\times I_{N_1}$ $\times J_1\times\cdots\times J_{N_2}$ tensor with entries

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$$(\mathcal{A}*_N\mathcal{B})_{i_1\cdots i_{N_1}j_1\cdots j_{N_2}}=\sum_{k_1\cdots k_N} a_{i_1\cdots i_{N_1}k_1\cdots k_N}b_{k_1\cdots k_N j_1\cdots j_{N_2}}.$$

Clearly, when $\mathcal{A}$ and $\mathcal{B}$ are matrices, the above product is the usual matrix product.

The Einstein product of tensors was widely used in the areas of the theory of relativity [5] and continuum mechanics [10], and the tensor systems via the Einstein product have many applications in continuum physics, engineering, isotropic and anisotropic elastic models [10]. In the process of solving a tensor equation via the Einstein product for the problems of high-dimensional PDEs and large discrete quantum models, an inverse of tensors via the Einstein product was proposed in [4]. For $\mathcal{A}\in\mathbb{C}^{I_1\times\cdots\times I_N\times I_1\times\cdots\times I_N}$, if there exists a tensor $\mathcal{X}$ such that $\mathcal{A}*_N\mathcal{X}=\mathcal{X}*_N\mathcal{A}=\mathcal{I}$, then $\mathcal{X}$ is called the inverse of $\mathcal{A}$. And $\mathcal{A}$ is called invertible, the inverse of $\mathcal{A}$ is denoted by $\mathcal{A}^{-1}$.

In order to give the minimum-norm least-square of tensor equations via the Einstein product, the $\{i\}$-inverse and Moore-Penrose inverse of tensors via the Einstein product were defined, and the representations for the Moore-Penrose inverse of some block tensors were established [18]. After that, the problem of inverse and generalized inverses of tensors via the Einstein product has attracted much attention including that the methods to compute the Moore-Penrose inverse of tensors [1], the perturbation theory for Moore-Penrose inverse [12], the properties of weighted Moore-Penrose inverse [8], the extreme learning machine based on Moore-Penrose inverse [7], and so on [14, 16, 17, 19–21]. Recently, the Drazin inverse of an even-order tensor via the Einstein product was proposed, and a singular tensor equation was studied [9].

For $\mathcal{A}\in\mathbb{C}^{I_1\times\cdots\times I_M\times J_1\times\cdots\times J_N}$,

$$\begin{aligned}\mathrm{R}(\mathcal{A})&=\left\{\mathcal{A}*_N\mathcal{X}:\mathcal{X}\in\mathbb{C}^{J_1\times\cdots\times J_N}\right\},\\ \mathrm{N}(\mathcal{A})&=\left\{\mathcal{X}\in\mathbb{C}^{J_1\times\cdots\times J_N}:\mathcal{A}*_N\mathcal{X}=0\right\}\end{aligned}$$

are called the range and the null space of $\mathcal{A}$, respectively. For a tensor $\mathcal{A}\in\mathbb{C}^{I_1\times\cdots\times I_N\times I_1\times\cdots\times I_N}$, $\mathcal{A}^k$ denotes the k power of $\mathcal{A}$, and $\mathcal{A}^0=\mathcal{I}$. The smallest nonnegative integer k such that $\mathrm{R}(\mathcal{A}^{k+1})=\mathrm{R}(\mathcal{A}^k)$ is called the index of $\mathcal{A}$, denoted by $\mathrm{ind}(\mathcal{A})$. For $\mathcal{A}\in\mathbb{C}^{I_1\times\cdots\times I_N\times I_1\times\cdots\times I_N}$, if there exists a tensor $\mathcal{X}\in\mathbb{C}^{I_1\times\cdots\times I_N\times I_1\times\cdots\times I_N}$ such that

$$\mathcal{A}^l*_N\mathcal{X}*_N\mathcal{A}=\mathcal{A}^l,\quad \mathcal{X}*_N\mathcal{A}*_N\mathcal{X}=\mathcal{X},\quad \mathcal{A}*_N\mathcal{X}=\mathcal{X}*_N\mathcal{A},$$

then $\mathcal{X}$ is called the Drazin inverse of $\mathcal{A}$, denoted by $\mathcal{X}=\mathcal{A}^D$, where $l\geqslant\mathrm{ind}(\mathcal{A})$. If $\mathrm{ind}(\mathcal{A})=1$, then $\mathcal{X}$ is called the group inverse of $\mathcal{A}$, denoted by $\mathcal{A}^\#$. We know that