Commun. Math. Res.
doi: 10.4208/cmr.2021-0040

Vol. **38**, No. 1, pp. 62-80
February 2022

Global Solutions of 3-D Inhomogeneous Navier-Stokes System with Large Viscosity in One Variable

Tiantian Hao*

Academy of Mathematics & Systems Science, Chinese Academy of Sciences, Beijing 100190, China, and School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing 100049, China.

Received 30 March 2021; Accepted 19 June 2021

Abstract. We consider the global well-posedness of three dimensional incompressible inhomogeneous Navier-Stokes equation with different viscous coefficients in the vertical and horizontal variables. In particular, when one of these viscous coefficients is large enough compared with the initial data and the initial density is close enough to a positive constant, we prove the global well-posedness of this system. This result extends the previous results in [9, 11] for the classical Navier-Stokes system.

AMS subject classifications: 35Q30, 76D03

Key words: Inhomogeneous Navier-Stokes system, anisotropic Littlewood-Paley theory, global well-posedness.

1 Introduction

We consider the global existence of strong solution to the following 3-D inhomogeneous incompressible anisotropic Navier-Stokes equations with initial density being sufficiently close to a positive constant in the critical space, $\dot{B}^{\frac{3}{p}}_{p,1}(\mathbb{R}^3)$ for some $p\in]3,4[$, and with a large viscous coefficient in one direction:

*Corresponding author. *Email address:* `htt@amss.ac.cn` (T. Hao)

$$\begin{cases} \partial_t\rho+u\cdot\nabla\rho=0, \quad (t,x)\in\mathbb{R}^+\times\mathbb{R}^3, \\ \rho(\partial_t u+u\cdot\nabla u)-\nu_{\rm h}\Delta_{\rm h}u-\nu_{\rm v}\partial_3^2 u+\nabla P=0, \\ \operatorname{div} u=0, \\ (\rho,u)|_{t=0}=(\rho_0,u_0), \end{cases} \tag{1.1}$$

where ρ,u stand for the density and velocity of the fluid respectively, ∇P is a scalar pressure function, which guarantees the divergence free condition of the velocity field, $\nu_{\rm h}$ and $\nu_{\rm v}$ are viscous coefficients so that $\nu_{\rm v}$ is much larger than $\nu_{\rm h}>0$, $\Delta_{\rm h}\stackrel{\rm def}{=}\partial_{x_1}^2+\partial_{x_2}^2$ designates the horizontal Laplacian. One may check [8] for more background of this system. The system (1.1) has three major basic features. First, the incompressibility expressed by fact that the vector field u is divergence free gives

$$\|\rho(t)\|_{L^\infty}=\|\rho_0\|_{L^\infty}.$$

Second, this system has the following enenrgy law

$$\frac{1}{2}\|\sqrt{\rho}u(t)\|_{L^2}^2+\nu_{\rm h}\|\nabla_{\rm h}u\|_{L_t^2(L^2)}^2+\nu_{\rm v}\|\partial_3 u\|_{L_t^2(L^2)}^2=\frac{1}{2}\|\sqrt{\rho_0}u_0\|_{L^2}^2. \tag{1.2}$$

The third basic feature is the scaling invariance property: If (ρ,u,P) is a solution of (1.1) on $[0,T]\times\mathbb{R}^3$, then the rescaled triplet $(\rho,u,P)_\lambda$ defined by

$$(\rho,u,P)_\lambda(t,x)\stackrel{\rm def}{=}\Big(\rho\big(\lambda^2t,\lambda x\big),\lambda u\big(\lambda^2t,\lambda x\big),\lambda^2P\big(\lambda^2t,\lambda x\big)\Big) \quad \text{with} \quad \lambda\in\mathbb{R}^+ \tag{1.3}$$

is also a solution of (1.1) on $[0,T/\lambda^2]\times\mathbb{R}^3$. Motivated by (1.3), Danchin [7] established the well-posedness of (1.1) in the so-called critical functional framework for small perturbations of some positive constant density.

When the density ρ is away from zero, we set $a\stackrel{\rm def}{=}\frac{1}{\rho}-1$. Then (1.1) can be equivalently formulated as

$$\begin{cases} \partial_t a+u\cdot\nabla a=0, \quad (t,x)\in\mathbb{R}^+\times\mathbb{R}^3, \\ \partial_t u+u\cdot\nabla u-\nu_{\rm h}(1+a)\Delta_{\rm h}u-\nu_{\rm v}(1+a)\partial_3^2u+(1+a)\nabla P=0, \\ \operatorname{div} u=0, \\ (a,u)|_{t=0}=(a_0,u_0). \end{cases} \tag{1.4}$$

One may check [5] and the references therein concerning the well-posedness theory of the system (1.1) or (1.4). In particular, when $\rho_0\in L^\infty$ with a positive lower bound and initial velocity being sufficiently small in the critical Besove space,