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Singularly Perturbed Renormalization Group Method and Its Significance in Dynamical Systems Theory

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Abstract. In this paper, we mainly investigate three topics on the renormalization group (RG) method to singularly perturbed problems: 1) We will present an explicit strategy of RG procedure to get the approximate solution up to any order. 2) We will refer that the RG procedure can, in fact, be used to get the normal form of differential dynamical systems. 3) We will also present the approximating center manifolds of the perturbed systems, and investigate the long time asymptotic behavior by means of RG formula.

AMS subject classifications: 34D15, 34E05, 37D10, 37G05

Key words: Renormalization group method, normal form, center manifolds.

1 Introduction

Renormalization group (RG) method in the singular perturbation theory was originally introduced by Chen *et al.* [1, 2] in 1980s, inspired from the classical renormalization idea in quantum mechanics [10]. The main goal of this method is to give a unified strategy to compute the effective approximate solution of different kinds of singular perturbation problems. So far, the RG method has been

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turned out to be very useful in a large number of singular perturbed problems, such as secular problems, center manifolds etc. [2,5,7,9,11,12,15,16,19].

Maybe the first rigorous investigation of RG method can be traced back to Ziane's consideration of following perturbed systems [22] in 1999:

$$\begin{cases} \dot{\mathbf{x}}+\dfrac{1}{\varepsilon}A\mathbf{x}=\mathbf{f}(\mathbf{x}), \\ \mathbf{x}(0)=\mathbf{x}_0, \end{cases} \tag{1.1}$$

where $\mathbf{x}\in\mathbb{C}^n$, ε is a small parameter, A is a complex diagonalizable matrix, and $\mathbf{f}(\mathbf{x})$ is a polynomial nonlinear term. Based on the typically renormalization procedure, Ziane obtained two approximate results under certain assumptions, the corresponding strategy is unified, concise and effective. Following Ziane's formulation, RG method has been successfully used to analyze a large kinds of problems including the singular perturbed semi-linear PDE problems [9,13,14,17,21].

In 2003, Temam and Wirosoetisno [20] considered a class of systems with the form

$$\begin{cases} \dot{\mathbf{x}}+\dfrac{1}{\varepsilon}L\mathbf{x}+A\mathbf{x}+B(\mathbf{x})=\mathbf{f}(t), \\ \mathbf{x}(0)=\mathbf{x}_0, \end{cases} \tag{1.2}$$

where L is a real antisymmetric matrix, A is a positive-definite matrix, $\mathbf{f}(t)$ is given with $|\mathbf{f}|_\infty=\operatorname{ess\,sup}_{0\le t\le\infty}|\mathbf{f}(t)|$ finite, and $B(\mathbf{x})=\sum_{i=1}^r B_i(\mathbf{x})$, with $B_i(\mathbf{x})$ the i-linear and completely antisymmetric in $\mathbf{x}$, i.e., the inner product $\langle B(\mathbf{x}),\mathbf{x}\rangle=0$ for all $\mathbf{x}\in\mathbb{R}^d$. The authors presented an implicit procedure to obtain the approximate solution up to any order, they also made several dynamical analysis about the conservation, or dissipation of energy in different cases, which implies the simplicity of RG method compared with other ones.

In 2008, Chiba [5] considered another class of singular perturbation problems as the following form:

$$\dot{\mathbf{x}}=\varepsilon\mathbf{g}(\mathbf{x},t,\varepsilon), \quad \mathbf{x}\in U, \tag{1.3}$$

where U is an open set in $\mathbb{C}^n$ and the closure $\bar{U}$ is compact, $\mathbf{g}(\mathbf{x},t,\varepsilon)$ is a vector field parameterized by $\varepsilon\in\mathbb{R}_+$. Inspired by the KBM theory, he presented a higher order RG theory for above system with a key assumption that the nonlinear terms are almost-periodic in t, and the set of corresponding Fourier exponents having no accumulation on $\mathbb{R}$. Moreover, his work turns out that, in many cases, RG theory can also lead to the existence of approximate invariant manifolds, inheritance of symmetries from those for the original equation to those for the RG equation, and unify traditional singular perturbation methods, such as the averaging method, the multiple time scale method and the center manifold reduction, etc.