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Some New Results on Purely Singular Splittings

Pingzhi Yuan*

School of Mathematical Science, South China Normal University, Guangzhou 510631, China.

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Abstract. Let G be a finite abelian group, M a set of integers and S a subset of G. We say that M and S form a splitting of G if every nonzero element g of G has a unique representation of the form $g=ms$ with $m\in M$ and $s\in S$, while 0 has no such representation. The splitting is called purely singular if for each prime divisor p of $|G|$, there is at least one element of M is divisible by p. In this paper, we continue the study of purely singular splittings of cyclic groups. We prove that if $k\geq 2$ is a positive integer such that $[-2k+1,2k+2]^*$ splits a cyclic group $\mathbb{Z}_m$, then $m=4k+2$. We prove also that if $M=[-k_1,k_2]^*$ splits $\mathbb{Z}_m$ purely singularly, and $15\leq k_1+k_2\leq 30$, then $m=1$, or $m=k_1+k_2+1$, or $k_1=0$ and $m=2k_2+1$.

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1 Introduction

Let G be a finite group, written additively, M a set of integers, and S a subset of G. We say that M and S form a splitting of G if every nonzero element g of G has a unique representation of the form $g=ms$ with $m\in M$ and $s\in S$, while 0 has no such representation. (Here ms denotes the sum of m s's if $m\geq 0$, and $-((-m)s)$ if $m<0$.) We write $G\setminus\{0\}=MS$ to indicate that M and S form a splitting of G. M is

*Corresponding author. *Email address:* `yuanpz@scnu.edu.cn` (P. Yuan)

referred to as the multiplier set and S as the splitter set. We also say that M splits G with a splitter set S, or simply that M splits G.

Let a,b be integers such that $a\le b$, denote

$$[a,b]=\{a,a+1,a+2,\ldots,b\},\quad [a,b]^*=\{a,a+1,a+2,\ldots,b\}\backslash\{0\}.$$

For any positive integer q, let $\mathbb{Z}_q$ be the ring of integers modulo q and $\mathbb{Z}_q^*=\mathbb{Z}_q\backslash\{0\}$. For $a\in\mathbb{Z}_q$, $o(a)$ denotes the order of a in the additive group $\mathbb{Z}_q$.

Let q be a positive integer and k_1,k_2 be non-negative integers with $0\le k_1\le k_2$. The set $B\subset\mathbb{Z}_q$ is called a splitter set (or a packing set) if all the sets

$$\{ab \pmod q): a\in[-k_1,k_2]\},\quad b\in B$$

have k_1+k_2 nonzero elements, and they are disjoint. We denote such a splitter set by $B[-k_1,k_2](q)$ set. A $B[-k_1,k_2](q)$ set of size n is called perfect if $n=\frac{q-1}{k_1+k_2}$. Clearly, a perfect set can exist only if $q\equiv 1 \pmod{k_1+k_2}$. A perfect $B[-k_1,k_2](q)$ set is called nonsingular if $\gcd(q,k_2!)=1$. Otherwise, the set is called singular. If for any prime $p|q$, there is some k with $0<k\le k_2$ such that $p|k$, then the perfect $B[-k_1,k_2](q)$ set is called purely singular.

Remark 1.1. Let q be a positive integer and k_1,k_2 be non-negative integers with $0\le k_1\le k_2$. Let $M=[-k_1,k_2]^*$. Then M splits an abelian group G of order q if and only if M splits $\mathbb{Z}_q$ by the works of Hamaker and Stein [1], Hickerson [2] and Schwarz [3]. It means that there is a subset $B\subset\mathbb{Z}_q$ such that B is a perfect $B[-k_1,k_2](q)$ sets for the cyclic group $\mathbb{Z}_q$. Therefore, we are only interested in considering purely singular perfect $B[-k_1,k_2](q)$ sets for the cyclic group $\mathbb{Z}_q$ and nonsingular perfect $B[-k_1,k_2](p)$ sets for an odd prime p.

In this paper, we focus our attention to the purely singular perfect $B[-k_1,k_2](q)$ sets for the cyclic group $\mathbb{Z}_q$. By taking as splitting set $S=\emptyset$, S=$\{1\}$, and $S=\{-1,1\}$, respectively, one see that $[1,k]$ splits $\mathbb{Z}_1$, $\mathbb{Z}_{k+1}$, and $\mathbb{Z}_{2k+1}$ for every k. It is conjectured that every purely singular splitting is one of these three types (see, e.g., [7]).

Conjecture 1.1. Let k be a positive integer. If $[1,k]$ splits the finite abelian group G purely singularly, then G is one of $\mathbb{Z}_1$, $\mathbb{Z}_{k+1}$, or $\mathbb{Z}_{2k+1}$.

Conjecture 1.1 has been verified by Hickerson for all $k<3000$ (see [7]).

As an analogy to the above conjecture, we have the following conjecture, which covers the related conjecture in [9].

Conjecture 1.2. Let k_1,k_2 be integers with $1\le k_1\le k_2$ and $k_1+k_2\ge 4$, then there does not exist any purely singular perfect $B[-k_1,k_2](m)$ set except for $m=1$ and $m=k_1+k_2+1$.