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Finite Geometry and Deep Holes of Reed-Solomon Codes over Finite Local Rings

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Abstract. In this paper, we first propose the maximum arc problem, normal rational curve conjecture, and extensions of normal rational curves over finite local rings, analogously to the finite geometry over finite fields. We then study the deep hole problem of generalized Reed-Solomon (RS) codes over finite local rings. Several different classes of deep holes are constructed. The relationship between finite geometry and deep holes of RS codes over finite local rings are also studied.

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1 Introduction

We first fix some notations which are valid for the whole paper.

- Let $\mathbb{A}$ be a finite local ring, $\mathbb{A}^\times$ be its unit group, and M be the unique maximal ideal.

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- Let $\mathbb{F}_q=\mathbb{A}/M$ be the residue field and σ be the projection from $\mathbb{A}$ to $\mathbb{A}/M$.
- Let $F=\{\alpha_1,\cdots,\alpha_q\}$ denote any set of complete representatives of $\mathbb{A}/M$. That is, $\mathbb{A}=\cup_{\alpha\in F}(\alpha+M)$.
- A subset $D=\{\alpha_1,\cdots,\alpha_n\}\subseteq\mathbb{A}$ is called substractive if $\alpha_i-\alpha_j\in\mathbb{A}^\times$ for all $i\neq j$.

1.1 Finite geometry over finite local rings

Analogously to the finite geometry over finite fields, we generalize the arcs and normal rational curves to finite local rings.

Recall that the ring $\mathbb{A}$ is called a local ring if $\mathbb{A}$ is commutative with a unit and has the unique maximal ideal M. A basic property of local ring $(\mathbb{A},M)$ is that $\mathbb{A}\setminus M$ consists of all the units of $\mathbb{A}$. The local ring is one of the most important objects in the study of algebraic geometry and commutative algebra [1].

Any two non-zero vectors $v,w\in\mathbb{A}^k$ is said to be equivalent if there exists $u\in\mathbb{A}^\times$ such that $v=uw$. And we denote it by $v\sim w$. Let $\mathbb{P}^{k-1}(\mathbb{A})=\mathbb{A}^k/\sim$ be the projective space of rank $k-1$ over $\mathbb{A}$. An n-arc $\chi\subset\mathbb{P}^{k-1}(\mathbb{A})$ is a subset of n points such that any k points in χ form a basis of $\mathbb{A}^k$. An n-arc is called complete if it is not a subset of any $(n+1)$-arc. Let $N_k(\mathbb{A})$ be the maximum possible value n such that there is an n-arc $\chi\subset\mathbb{P}^{k-1}(\mathbb{A})$. For any integer $2\leq k\leq q$ and $\alpha\in\mathbb{A}\cup\infty$, we define

$$c_k(\alpha)=\begin{cases}\left(1,\alpha,\alpha^2,\cdots,\alpha^{k-1}\right)^T, & \text{if} \quad \alpha\in\mathbb{A},\\ (0,0,\cdots,0,1)^T, & \text{if} \quad \alpha=\infty.\end{cases}$$

For any substractive subset $D\subset\mathbb{A}\cup\infty$, $\mathcal{C}_k(D)=\{c_k(\alpha)\,|\,\alpha\in D\}$ is called a normal rational curve (NRC) associated to D, which is obviously an arc.

As a generalization of the classical MDS conjecture over finite fields, we have the following conjecture.

Conjecture 1.1. Let $2\leq k\leq q$. $N_k(\mathbb{A})=q+1$ with exception $q=2^m$ and $k\in\{3,q-1\}$ when $N_k(\mathbb{A})=q+2$.

Since NRCs associated to substractive subsets are obviously arcs, we have the following much weaker conjecture.

Conjecture 1.2. For any integer $2\leq k\leq q$ ($k\notin\{3,q-1\}$ if q is even), the NRC $\mathcal{C}_k(F\cup\infty)$ is complete.

We will show that the above two conjectures are equivalent to the corresponding conjectures over finite fields. Another interesting problem is when the extension of an NRC is an arc.